Equilibria of 'discrete' integrable systems and deformation of classical orthogonal polynomials

2004 J. Phys. A: Math. Gen. 37 11841

(http://iopscience.iop.org/0305-4470/37/49/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 11841–11876 PII: S0305-4470(04)83968-4

Equilibria of 'discrete' integrable systems and deformation of classical orthogonal polynomials

S Odake[1] **and R Sasaki**[2]

[1] Department of Physics, Shinshu University, Matsumoto 390-8621, Japan
[2] Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

Received 23 July 2004, in final form 20 October 2004
Published 24 November 2004
Online at stacks.iop.org/JPhysA/37/11841
doi:10.1088/0305-4470/37/49/006

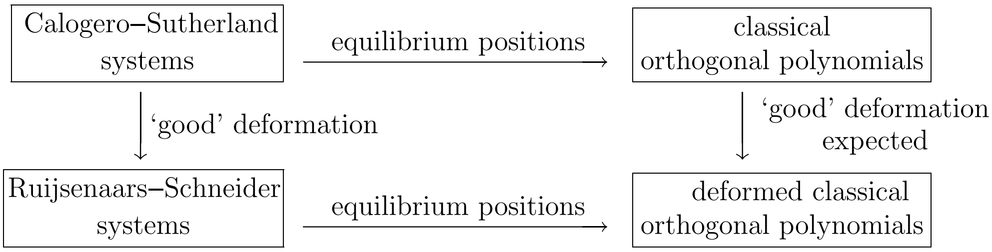
Abstract
The Ruijsenaars–Schneider systems are 'discrete' version of the Calogero–Moser (C–M) systems in the sense that the momentum operator p appears in the Hamiltonians as a polynomial in $e^{\pm\beta' p}$ (β' is a deformation parameter) instead of an ordinary polynomial in p in the hierarchies of C–M systems. We determine the polynomials describing the equilibrium positions of the rational and trigonometric Ruijsenaars–Schneider systems based on classical root systems. These are deformations of the classical orthogonal polynomials, the Hermite, Laguerre and Jacobi polynomials which describe the equilibrium positions of the corresponding Calogero and Sutherland systems. The orthogonality of the original polynomials is inherited by the deformed ones which satisfy three-term recurrence and certain functional equations. The latter reduce to the celebrated second-order differential equations satisfied by the classical orthogonal polynomials.

PACS numbers: 02.20.−a, 02.30.Gp, 02.30.Ik

1. Introduction

Exactly solvable or quasi-exactly solvable multi-particle quantum mechanical systems have many remarkable properties. By definition, the entire (or a part of the) spectrum and the corresponding eigenfunctions are calculable by algebraic means. The corresponding classical systems also share many 'quantum' features. For example, the frequencies of small oscillations near the classical equilibrium are 'quantized' together with the eigenvalues of the associated Lax matrices at the equilibrium. These phenomena have been explored extensively for multi-particle dynamics based on root systems, in particular, for the Calogero and Sutherland systems [1–3] by Corrigan–Sasaki [4]. Similar phenomena are also reported by Ragnisco–Sasaki [5] for Ruijsenaars–Schneider systems [6–9], which are *deformations* of C–M systems.

In this paper we will discuss one special aspect of the classical equilibria of exactly solvable systems based on classical root systems, the rational and trigonometric Ruijsenaars–Schneider systems. Namely, the determination of the equilibrium positions and their description in terms of certain polynomials. It is known that for the Calogero and Sutherland systems, the equilibrium positions are described by the zeros of the classical orthogonal polynomials, i.e. the Hermite, Laguerre, Chebyshev, Legendre, Gegenbauer and Jacobi polynomials [4, 10, 11]. The Ruijsenaars–Schneider systems are 'good' deformation of the Calogero and Sutherland systems. Here is one interesting piece of evidence. It was known [12] that the singular vectors of the Virasoro and W_N algebras, in the free field representation, are related to Jack polynomials [13], the quantum eigenfunctions of the A type Sutherland systems. The deformed Virasoro and W_N algebras were discovered by using the relation between the Sutherland system and the trigonometric Ruijsenaars–Schneider system of the A type root system [14]. Therefore, it is expected that equilibrium positions of the rational and trigonometric Ruijsenaars–Schneider systems would give certain 'good' deformation of the classical orthogonal polynomials:



In the Ragnisco–Sasaki paper [5], based on numerical analysis, the explicit forms of the lower degree members of the one-parameter deformation of the Hermite and Laguerre polynomials were presented. The present authors continued the numerical analysis and obtained the explicit forms of the lower degree members of the one and/or two-parameter deformation of the Hermite polynomial, one-, two- and/or three-parameter deformation of the Laguerre polynomial, one-parameter deformation of the Jacobi (and Gegenbauer and Legendre) polynomials. They are also polynomials, or rational functions in the deformation parameter(s) with *integer coefficients*.

Remarkably, the orthogonality of the original polynomials is inherited by the deformed ones. The equations determining the equilibrium can be reformulated as functional equations determining the polynomials. These functional equations are difference analogues of the celebrated second-order differential equations satisfied by the classical orthogonal polynomials. Three-term recurrence for the deformed polynomials, the necessary and sufficient condition for orthogonality, can be derived from these functional equations. Dynamical stability of the Hamiltonian system, or bounded-from-belowness of its potentials, is achieved by restricting the parameter space of the coupling constants, usually by *positive coupling constants*, which in turn guarantees the positive definiteness of the inner product governing the orthogonal polynomials, the deformed as well as undeformed. These deformed polynomials are *not* the so-called q-deformed versions of the above classical polynomials [15].

This paper is organized as follows. In section 2, first we recall the essence of the Calogero–Sutherland systems and their equilibria, which are described by the Hermite, Laguerre and Jacobi polynomials. Next the Hamiltonians and potentials of the Ruijsenaars–Schneider (R–S) systems are briefly recapitulated, and two types of the rational systems and one trigonometric systems for the classical root systems are introduced. Then the equations for their equilibrium positions are written. For later use we review the relation between the orthogonal polynomials and the three-term recurrence. Sections 3–5 give the main results of this paper. In section 3, we determine the equilibrium positions of the rational R–S systems for the A type root system and the deformation of the Hermite polynomial is presented. For one-parameter deformation case, we derive the explicit forms of the generating function and the weight function of the inner product. In section 4, equilibrium positions of the rational R–S systems for the B, C, D, BC type root system are determined and the deformation of the Laguerre polynomial is presented together with the explicit forms of the functional equations and three-term recurrence. In section 5, equilibrium positions of the trigonometric R–S systems for the B, C, D, BC type root system are determined and the deformation of the Jacobi (and Gegenbauer) polynomial is presented together with the explicit forms of the functional equations and three-term recurrence. Classical orthogonal polynomials, e.g., the Hermite and Laguerre, satisfy many interesting identities among themselves. Those having a root theoretic explanation (*folding*) are shown to be preserved after deformation. The final section is devoted to a summary and comments. Identification of the deformed orthogonal polynomials within the so-called Askey-scheme of hypergeometric orthogonal polynomials [16] is reported here. The relation between the functional equation and the three-term recurrence is discussed in the appendix.

2. Potentials and equilibrium positions

In this section, we set up models and present the equations for their equilibrium positions. We consider a multi-particle classical mechanics governed by a classical Hamiltonian $H(p, q)$. The dynamical variables are the coordinates $\{q_j | j = 1, \ldots, r\}$ and their canonically conjugate momenta $\{p_j | j = 1, \ldots, r\}$. These will be denoted by vectors in $\mathbb{R}^r$

$$q = {}^t(q_1, \ldots, q_r), \qquad p = {}^t(p_1, \ldots, p_r),$$

in which r is the number of particles (and it is also the rank of the underlying root system Δ except for the A case). The canonical equations of motion are

$$\dot{q}_j = \frac{\partial H(p, q)}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H(p, q)}{\partial q_j}. \tag{2.1}$$

The equilibrium positions are the stationary solution

$$p = 0, \qquad q = \bar{q}, \tag{2.2}$$

in which $\bar{q}$ satisfies

$$\left.\frac{\partial H(0, q)}{\partial q_j}\right|_{q=\bar{q}} = 0 \qquad (j = 1, \ldots, r). \tag{2.3}$$

We will discuss Ruijsenaars-type models associated with the *classical* root systems, namely the A_{r-1}, B_r, C_r, D_r and BC_r. The fact that all the roots of the classical root systems are neatly expressed in terms of the orthonormal basis of $\mathbb{R}^r$ makes formulation much simpler than those systems based on the exceptional root systems. The sets of positive roots of the classical root systems are

$$\begin{aligned}
&A_{r-1}\colon\quad \Delta_+ = \{e_j - e_k | 1 \leqslant j < k \leqslant r\},\\
&B_r\colon\quad \Delta_{L+} = \{e_j \pm e_k | 1 \leqslant j < k \leqslant r\}, \qquad \Delta_{S+} = \{e_j | 1 \leqslant j \leqslant r\},\\
&C_r\colon\quad \Delta_{S+} = \{e_j \pm e_k | 1 \leqslant j < k \leqslant r\}, \qquad \Delta_{L+} = \{2e_j | 1 \leqslant j \leqslant r\},\\
&D_r\colon\quad \Delta_+ = \{e_j \pm e_k | 1 \leqslant j < k \leqslant r\},\\
&BC_r\colon\quad \Delta_{M+} = \{e_j \pm e_k | 1 \leqslant j < k \leqslant r\},\\
&\qquad\quad \Delta_{S+} = \{e_j | 1 \leqslant j \leqslant r\}, \qquad \Delta_{L+} = \{2e_j | 1 \leqslant j \leqslant r\},
\end{aligned}$$

where $\{e_j\}$ is an orthonormal basis of $\mathbb{R}^r$. The subscripts L, M and S stand for long, middle and short roots, respectively.

It is well known that the non-simply laced root systems are obtained from simply laced ones by *folding*. In the present case, the relevant ones are

$$A_{2r-1} \to C_r, \qquad D_{r+1} \to B_r, \qquad A_{2r} \to BC_r. \tag{2.4}$$

At the level of the dynamical variables and Hamiltonians, the above foldings are realized as

$$A_{2r-1} \to C_r: \quad p_{2r+1-j} = -p_j, \qquad q_{2r+1-j} = -q_j \qquad (j = 1, \ldots, r), \tag{2.5}$$

$$D_{r+1} \to B_r: \quad p_{r+1} = q_{r+1} = 0, \tag{2.6}$$

$$A_{2r} \to BC_r: \quad p_{2r+2-j} = -p_j, \qquad q_{2r+2-j} = -q_j \qquad (j = 1, \ldots, r),$$
$$p_{r+1} = q_{r+1} = 0. \tag{2.7}$$

2.1. Calogero and Sutherland systems

For later comparison, we summarize the Calogero and Sutherland systems associated with the *classical* root systems only, namely the A_{r-1}, B_r, C_r, D_r and BC_r.

The Hamiltonian of the classical Calogero and Sutherland systems is

$$H_{\mathrm{CS}}(p, q) = \frac{1}{2} \sum_{j=1}^{r} p_j^2 + V_{\mathrm{C}}(q), \tag{2.8}$$

where the classical potential V_{C} is given below explicitly. For all cases this classical potential V_{C} can be written in terms of the *prepotential* $W(q)$ [17]

$$V_{\mathrm{C}}(q) = \frac{1}{2} \sum_{j=1}^{r} \left(\frac{\partial W(q)}{\partial q_j} \right)^2. \tag{2.9}$$

The equations for the equilibrium positions (2.3) reduce to the following equations:

$$\left. \frac{\partial W(q)}{\partial q_j} \right|_{q=\bar{q}} = 0 \qquad (j = 1, \ldots, r). \tag{2.10}$$

2.1.1. Calogero systems.

The classical potential V_{C} and prepotential W are

$$V_{\mathrm{C}}(q) = \frac{\omega^2}{2} \sum_{j=1}^{r} q_j^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} \frac{g_\rho^2 \rho^2}{(\rho \cdot q)^2}, \tag{2.11}$$

$$W(q) = -\frac{\omega}{2} \sum_{j=1}^{r} q_j^2 + \sum_{\rho \in \Delta_+} g_\rho \log|(\rho \cdot q)|, \tag{2.12}$$

where ω is the (positive) frequency of the harmonic confining potential and g_ρ are real positive coupling constants depending on the length of the roots. They are: one coupling g for all roots for the A_{r-1} and D_r, two independent couplings g_L and g_S for B_r and C_r corresponding to the long and short roots, respectively. These conventions are the same for all other types of potentials considered in this paper. For example, the C_r model is

$$C_r:\quad V_{\rm C}(q)=\frac{\omega^2}{2}\sum_{j=1}^{r}q_j^2+\frac{g_S^2}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{r}\left(\frac{1}{(q_j-q_k)^2}+\frac{1}{(q_j+q_k)^2}\right)+\frac{g_L^2}{2}\sum_{j=1}^{r}\frac{1}{q_j^2},$$

$$W(q)=-\frac{\omega}{2}\sum_{j=1}^{r}q_j^2+\frac{g_S}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{r}\log\left|q_j^2-q_k^2\right|+g_L\sum_{j=1}^{r}\log|2q_j|.$$

There is no distinction between the rational B_r and C_r models because of the replacement $g_S\leftrightarrow g_L$. The D_r model can be considered as a special case of the B_r with $g_L=g$ and $g_S=0$.

The systems obtained by *folding* (2.5)–(2.7) have a special ratio of couplings. They are

$$\text{folded } C_r:\quad (g_L,g_S)=\left(\tfrac{1}{2},1\right)g,\qquad \text{folded } B_r:\quad (g_L,g_S)=(1,2)g,$$
$$\text{folded } BC_r:\quad (g_L,g_M,g_S)=\left(\tfrac{1}{2},1,1\right)g. \tag{2.13}$$

The equations for the equilibrium position (2.10) are

$$A_{r-1}:\quad \sum_{\substack{k=1\\ k\neq j}}^{r}\frac{1}{\bar q_j-\bar q_k}=\frac{\omega}{g}\bar q_j, \tag{2.14}$$

$$B_r:\quad \sum_{\substack{k=1\\ k\neq j}}^{r}\frac{2\bar q_j}{\bar q_j^2-\bar q_k^2}=\frac{\omega}{g_L}\bar q_j-\frac{g_S}{g_L}\frac{1}{\bar q_j}, \tag{2.15}$$

$$C_r:\quad \sum_{\substack{k=1\\ k\neq j}}^{r}\frac{2\bar q_j}{\bar q_j^2-\bar q_k^2}=\frac{\omega}{g_S}\bar q_j-\frac{g_L}{g_S}\frac{1}{\bar q_j}, \tag{2.16}$$

$$D_r:\quad \sum_{\substack{k=1\\ k\neq j}}^{r}\frac{2\bar q_j}{\bar q_j^2-\bar q_k^2}=\frac{\omega}{g}\bar q_j. \tag{2.17}$$

They determine the zeros of the Hermite and Laguerre polynomials. In other words, if we define $\bar q_j=\sqrt{\frac{g}{\omega}}y_j$ for A_{r-1}, then the polynomial having $\{y_j\}$ as zeros is the Hermite polynomial [10, 18, 4]:

$$2^r\prod_{j=1}^{r}(x-y_j)=r!\sum_{j=0}^{[\frac{r}{2}]}\frac{(-1)^j(2x)^{r-2j}}{j!(r-2j)!}\stackrel{\rm def}{=}H_r(x). \tag{2.18}$$

For the C_r (or B_r) model let us define $\bar q_j=\sqrt{\frac{g_S}{\omega}}y_j$, $\alpha=\frac{g_L}{g_S}-1$ then having $\left\{y_j^2\right\}$ as zeros is the Laguerre polynomial [10, 18, 4]:

$$\frac{(-1)^r}{r!}\prod_{j=1}^{r}\left(x-y_j^2\right)=\sum_{j=0}^{r}\binom{r+\alpha}{r-j}\frac{(-x)^j}{r!}\stackrel{\rm def}{=}L_r^{(\alpha)}(x). \tag{2.19}$$

For the D_r root system, it is the Laguerre polynomial $L_r^{(-1)}(x)$.

The identities between the Hermite and Laguerre polynomials

$$2^{-2r}H_{2r}(x) = (-1)^r r! L_r^{(-\frac{1}{2})}(x^2), \tag{2.20}$$

$$2^{-2r-1}H_{2r+1}(x) = x(-1)^r r! L_r^{(\frac{1}{2})}(x^2), \tag{2.21}$$

are well known. The former identity (2.20) for the even degree Hermite polynomial can be understood as a consequence of the folding of the root system $A_{2r-1} \to C_r$, see (2.5). Likewise the latter identity (2.21) for the odd degree Hermite polynomial can be understood as a consequence of the folding of the root system $A_{2r} \to BC_r$, see (2.7). Next let us consider the folding $D_{r+1} \to B_r$ (2.6), which leads to the identity [4] among the Laguerre polynomials of different indices:

$$(r+1)L_{r+1}^{(-1)}(x) = -xL_r^{(1)}(x). \tag{2.22}$$

We will see that these identities (2.20), (2.21) and (2.22) are also nicely *deformed* with one parameter (4.46), (4.48) and (4.50) and with two parameters (4.47), (4.49) and (4.51).

2.1.2. Sutherland systems. The classical potential $V_{\rm C}$ and prepotential W are (except for $V_{\rm C}$ of BC_r)

$$V_{\rm C}(q) = \frac{1}{2}\sum_{\rho\in\Delta_+}\frac{g_\rho^2\rho^2}{\sin^2(\rho\cdot q)}, \tag{2.23}$$

$$W(q) = \sum_{\rho\in\Delta_+} g_\rho \log|\sin(\rho\cdot q)|, \tag{2.24}$$

where g_ρ are real positive coupling constants. The classical potential $V_{\rm C}$ of the BC_r model is given by

$$BC_r:\quad V_{\rm C}(q) = \frac{g_M^2}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{r}\left(\frac{1}{\sin^2(q_j-q_k)}+\frac{1}{\sin^2(q_j+q_k)}\right) + 2g_L^2\sum_{j=1}^{r}\frac{1}{\sin^2 2q_j}$$
$$+\frac{g_S(g_S+2g_L)}{2}\sum_{j=1}^{r}\frac{1}{\sin^2 q_j}. \tag{2.25}$$

The $B_r(C_r)$ potential is obtained by setting $g_L = 0, g_M \to g_L$ ($g_S = 0, g_M \to g_S$).

The equations for the equilibrium position (2.10) are

$$A_{r-1}:\quad \sum_{\substack{k=1\\ k\neq j}}^{r}\cot(\bar{q}_j-\bar{q}_k) = 0, \tag{2.26}$$

$$B_r:\quad \sum_{\substack{k=1\\ k\neq j}}^{r}(\cot(\bar{q}_j-\bar{q}_k)+\cot(\bar{q}_j+\bar{q}_k)) = -\frac{g_S}{g_L}\cot\bar{q}_j, \tag{2.27}$$

$$C_r:\quad \sum_{\substack{k=1\\ k\neq j}}^{r}(\cot(\bar{q}_j-\bar{q}_k)+\cot(\bar{q}_j+\bar{q}_k)) = -2\frac{g_L}{g_S}\cot 2\bar{q}_j, \tag{2.28}$$

$$BC_r: \quad \sum_{\substack{k=1\\k\neq j}}^{r}(\cot(\bar{q}_j-\bar{q}_k)+\cot(\bar{q}_j+\bar{q}_k)) = -\frac{g_S}{g_M}\cot\bar{q}_j - 2\frac{g_L}{g_M}\cot 2\bar{q}_j, \tag{2.29}$$

$$D_r: \quad \sum_{\substack{k=1\\k\neq j}}^{r}(\cot(\bar{q}_j-\bar{q}_k)+\cot(\bar{q}_j+\bar{q}_k)) = 0. \tag{2.30}$$

The equilibrium positions of the A_{r-1} model are related to the Chebyshev polynomial and those of the other models are related to the Jacobi polynomials.

For the A_{r-1}, the equilibrium positions are '*equally spaced*' and translational invariant,

$$\bar{q} = \frac{\pi}{r}{}^t(r, r-1, \ldots, 1) + \xi^t(1, 1, \ldots, 1), \qquad \xi \in \mathbb{R}: \text{arbitrary}. \tag{2.31}$$

We choose this constant shift ξ such that the 'centre of mass' coordinate vanishes, $\sum_{j=1}^{r}\bar{q}_j = 0$:

$$\bar{q}_j = \frac{\pi(r+1-j)}{r} - \frac{\pi(r+1)}{2r} = \frac{\pi}{2} - \frac{\pi(2j-1)}{2r} = -\bar{q}_{r+1-j} \qquad (j = 1, \ldots, r). \tag{2.32}$$

Then the degree r (the dimension of the vector representation) polynomial in x, having zeros at $\{\sin\bar{q}_j\}$,

$$2^{r-1}\prod_{j=1}^{r}(x-\sin\bar{q}_j) = 2^{r-1}\prod_{j=1}^{r}\left(x-\cos\frac{\pi(2j-1)}{2r}\right) \stackrel{\text{def}}{=} T_r(x), \tag{2.33}$$

is the Chebyshev polynomial of the first kind, $T_n(\cos\varphi) = \cos(n\varphi)$.

For the solution $\{\bar{q}_j\}$ of the BC_r (2.29), $\cos 2\bar{q}_j$ is the zero of the Jacobi polynomial $P_r^{(\alpha,\beta)}(x)$ with $\alpha = \frac{g_S}{g_M} + \frac{g_L}{g_M} - 1$ and $\beta = \frac{g_L}{g_M} - 1$,

$$2^{-r}\binom{\alpha+\beta+2r}{r}\prod_{j=1}^{r}(x-\cos 2\bar{q}_j) = \sum_{j=0}^{r}\binom{\alpha+r}{r-j}\binom{\alpha+\beta+r+j}{j}2^{-j}(x-1)^j \stackrel{\text{def}}{=} P_r^{(\alpha,\beta)}(x). \tag{2.34}$$

It is easily shown that $\bar{q}'_j = \frac{\pi}{2} - \bar{q}_j$ satisfies (2.29) with $\alpha \leftrightarrow \beta$, which implies $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$. For the solution $\{\bar{q}_j\}$ of the C_r (2.28), $\cos 2\bar{q}_j$ is the zero of the Gegenbauer polynomial $C_r^{(\alpha+\frac{1}{2})}(x)$ with $\alpha = \frac{g_L}{g_S} - 1$,

$$\begin{aligned} 2^r\binom{\alpha-\frac{1}{2}+r}{r}&\prod_{j=1}^{r}(x-\cos 2\bar{q}_j) \\ &= \binom{2\alpha+r}{r}\binom{\alpha+r}{r}^{-1}\sum_{j=0}^{r}\binom{\alpha+r}{r-j}\binom{\alpha+\beta+r+j}{j}2^{-j}(x-1)^j \stackrel{\text{def}}{=} C_r^{(\alpha+\frac{1}{2})}(x). \end{aligned} \tag{2.35}$$

This is a special case of $P_r^{(\alpha,\beta)}(x)$ with another normalization,

$$C_r^{(\alpha+\frac{1}{2})}(x) = \binom{2\alpha+r}{r}\binom{\alpha+r}{r}^{-1}P_r^{(\alpha,\alpha)}(x). \tag{2.36}$$

For the solution $\{\bar{q}_j\}$ of the B_r (2.27), $\cos 2\bar{q}_j$ is the zero of $P_r^{(\alpha,-1)}(x)$ with $\alpha = \frac{g_S}{g_L} - 1$. For the solution $\{\bar{q}_j\}$ of the D_r (2.30), $\cos 2\bar{q}_j$ is the zero of $P_r^{(-1,-1)}(x)$.

The known identities between the Chebyshev and Jacobi polynomials and between the Jacobi polynomials can be understood as consequences of the *folding*:

$$A_{2r-1} \to C_r: \quad 2^{1-2r} T_{2r}(x) = (-1)^r \binom{2r-1}{r}^{-1} P_r^{(-\frac{1}{2},-\frac{1}{2})}(1-2x^2), \tag{2.37}$$

$$A_{2r} \to BC_r: \quad 2^{-2r} T_{2r+1}(x) = x(-1)^r \binom{2r}{r}^{-1} P_r^{(\frac{1}{2},-\frac{1}{2})}(1-2x^2), \tag{2.38}$$

$$D_{r+1} \to B_r: \quad 2(r+1) P_{r+1}^{(-1,-1)}(x) = r(x-1) P_r^{(1,-1)}(x). \tag{2.39}$$

We will see in the following that the first two identities are not deformed but the third one is nicely deformed (5.53).

The Gegenbauer polynomial and the Jacobi polynomial are also related by the quadratic transformations:

$$C_{2n}^{(\alpha+\frac{1}{2})}(x) = 2^{2n} \binom{\alpha-\frac{1}{2}+n}{n} \binom{2n}{n}^{-1} P_n^{(\alpha,-\frac{1}{2})}(2x^2-1), \tag{2.40}$$

$$C_{2n+1}^{(\alpha+\frac{1}{2})}(x) = 2^{2n+1} \binom{\alpha+\frac{1}{2}+n}{n+1} \binom{2n+1}{n}^{-1} x P_n^{(\alpha,\frac{1}{2})}(2x^2-1). \tag{2.41}$$

However, these identities do not seem to have a folding type explanation. Indeed, the deformed Gegenbauer polynomials $C_n^{(\alpha+\frac{1}{2})}(x,\delta)$ (2.35), $\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta)$ (5.26) and the deformed Jacobi polynomial $P_n^{(\alpha,\beta)}(x,\delta)$ (5.2) do not satisfy this type of identities for generic δ.

2.2. Ruijsenaars-type systems

Following Ruijsenaars–Schneider [6] and van Diejen [7], the Hamiltonian of the Ruijsenaars systems is (the deformation parameter β' of $e^{\pm\beta' p}$ is set to unity, $\beta' = 1$)

$$H(p,q) = \sum_{j=1}^{r} \left(\cosh p_j \sqrt{V_j(q) V_j^*(q)} - \frac{1}{2}(V_j(q) + V_j^*(q)) \right). \tag{2.42}$$

The form of the function $V_j = V_j(q)$ and its complex conjugate V_j^* are determined by the root system Δ as

$$A_{r-1}: \quad V_j(q) = w(q_j) \prod_{\substack{k=1\\k\neq j}}^{r} v(q_j - q_k) \qquad (j = 1, \ldots, r), \tag{2.43}$$

$$B_r, C_r, D_r, BC_r: \quad V_j(q) = w(q_j) \prod_{\substack{k=1\\k\neq j}}^{r} v(q_j - q_k) v(q_j + q_k) \qquad (j = 1, \ldots, r). \tag{2.44}$$

The elementary potential functions v and w depend on the nature of interactions (rational, trigonometric, etc) and the root system Δ. Their explicit forms will be given below. When V satisfies the simple identity $\sum_j (V_j(q) + V_j^*(q)) = \text{const}$, the Hamiltonian (2.42) could be replaced by a simpler one

$$H'(p,q) = \sum_{j=1}^{r} \cosh p_j \sqrt{V_j(q) V_j^*(q)}, \tag{2.45}$$

which is obviously positive definite and usually used as a starting point for the trigonometric (hyperbolic) interaction theory.

The above Hamiltonian (2.42) is a hyperbolic function of the momentum operator p instead of a polynomial in the hierarchy of C–M systems or other ordinary dynamical systems. In quantum theoretical setting this Hamiltonian causes finite shifts of the wavefunction in the imaginary direction, i.e. $\cosh p\psi(q) = \frac{1}{2}(\psi(q - i\hbar) + \psi(q + i\hbar))$, in which $\hbar$ is Planck's constant. This is why the R–S systems are sometimes called 'discrete' dynamical systems.

The equation (2.3) of equilibrium positions (2.2) can be simplified in the following way. By expanding the Hamiltonian around the stationary solution (2.2), we obtain

$$H(p,q) = K(p) + P(q) + \text{higher order terms in } p, \qquad K(p) = \frac{1}{2}\sum_{j=1}^{r}|V_j(\bar{q})|p_j^2, \tag{2.46}$$

and the 'potential' P is given by

$$P(q) = \sum_{j=1}^{r}\left(\sqrt{V_j(q)V_j^*(q)} - \frac{1}{2}(V_j(q) + V_j^*(q))\right) = -\frac{1}{2}\sum_{j=1}^{r}\left(\sqrt{V_j(q)} - \sqrt{V_j^*(q)}\right)^2. \tag{2.47}$$

This should be compared with the classical potential in the Calogero–Sutherland systems (2.9). It is obvious that the equilibrium is achieved at the point(s) in which all the functions V_j become *real* and *positive*:

$$V_j(\bar{q}) = V_j^*(\bar{q}) > 0 \qquad (j = 1, 2, \ldots, r). \tag{2.48}$$

The equilibrium point is the absolute minimum of the potential P. The system of equations (2.48) is invariant under any permutation of $\{\bar{q}_j\}$. For v and w considered in this paper, they are also invariant under the transformation $q \to q' = -q$. Except for the A case, they are also invariant under the transformation $q \to q' = {}^t(\epsilon_1 q_1, \ldots, \epsilon_r q_r)$ with $\epsilon_i = \pm 1$.

The functions v and w considered in this paper have properties

$$v(-x) = v^*(x), \qquad w(-x) = w^*(x), \tag{2.49}$$

which allow the *folding* (2.5)–(2.7) of the Ruijsenaars-type Hamiltonians:

$$H^{A_{2r-1}}(p,q)|_{\substack{p_{2r+1-j}=-p_j\\ q_{2r+1-j}=-q_j}} = 2\tilde{H}(p,q), \qquad \tilde{v}(x) = v^A(x), \qquad \tilde{w}(x) = w^A(x)v^A(2x), \tag{2.50}$$

$$H^{A_{2r}}(p,q)|_{\substack{p_{2r+2-j}=-p_j\\ q_{2r+2-j}=-q_j}} = 2\tilde{H}(p,q), \qquad \tilde{v}(x) = v^A(x), \qquad \tilde{w}(x) = w^A(x)v^A(x)v^A(2x), \tag{2.51}$$

$$H^{D_{r+1}}(p,q)|_{p_{r+1}=0,q_{r+1}=0} = \tilde{H}(p,q), \qquad \tilde{v}(x) = v^D(x), \qquad \tilde{w}(x) = w^D(x)v^D(x)^2, \tag{2.52}$$

where $\tilde{H}$ is

$$\begin{aligned}\tilde{H}(p,q) &= \sum_{j=1}^{r}\left(\cosh p_j\sqrt{\tilde{V}_j(q)\tilde{V}_j^*(q)} - \frac{1}{2}(\tilde{V}_j(q) + \tilde{V}_j^*(q))\right),\\ \tilde{V}_j(q) &= \tilde{w}(q_j)\prod_{\substack{k=1\\ k\neq j}}^{r}\tilde{v}(q_j - q_k)\tilde{v}(q_j + q_k).\end{aligned} \tag{2.53}$$

The folded systems (2.50), (2.51) and (2.52) correspond to the folding $A_{2r-1} \to C_r$, $A_{2r} \to BC_r$ and $D_{r+1} \to B_r$, respectively. The coupling constants in these folded systems are not independent as shown in (2.13).

2.2.1. Ruijsenaars–Calogero systems. The first example to be discussed is the 'discrete' analogue of the Calogero systems [1], to be called the Ruijsenaars–Calogero systems, which were introduced by van Diejen for the classical root systems only [7, 8]. The original Calogero systems [1] have the rational ($1/(\text{distance})^2$) potential plus the harmonic confining potential, having two coupling constants g and ω for the systems based on the simply-laced root systems, A and D, and three couplings ω and g_L for the long roots and g_S for the short roots in the B and C root systems.

Two varieties (deformation) of 'discrete' Calogero systems are known. The first has two (three for the non-simply-laced root systems) coupling constants g (g_L and g_S) and a which corresponds to ω in the Calogero systems. The second has three (four for the non-simply-laced root systems) coupling constants g (g_L and g_S) and a, b both of which correspond to ω. The integrability (classical and quantum) of these systems was discussed by van Diejen in some detail [7, 8]. The former can be considered as a limiting case ($b \to \infty$) of the latter.

Linear confining potential case. The dynamical system is defined by giving the explicit forms of the elementary potential functions v and w. For the simply-laced root systems A and D they are

$$A, D: \quad v(x) = 1 - \mathrm{i}\frac{g}{x}, \qquad w(x) = 1 + \mathrm{i}\frac{x}{a}, \tag{2.54}$$

in which a and g are real positive coupling constants. For the non-simply-laced root systems B, C and $\widetilde{BC}$, we have

$$B: \quad v(x) = 1 - \mathrm{i}\frac{g_L}{x}, \qquad w(x) = \left(1 + \mathrm{i}\frac{x}{a}\right)\left(1 - \mathrm{i}\frac{g_S}{2x}\right)^2, \tag{2.55}$$

$$C: \quad v(x) = 1 - \mathrm{i}\frac{g_S}{x}, \qquad w(x) = \left(1 + \mathrm{i}\frac{x}{a}\right)\left(1 - \mathrm{i}\frac{g_L}{x}\right), \tag{2.56}$$

$$\widetilde{BC}: \quad v(x) = 1 - \mathrm{i}\frac{g_0}{x}, \qquad w(x) = \left(1 + \mathrm{i}\frac{x}{a}\right)\left(1 - \mathrm{i}\frac{g_1}{x}\right)\left(1 - \mathrm{i}\frac{g_2}{x}\right), \tag{2.57}$$

in which $a, g_L, g_S, g_0, g_1, g_2$ are *independent* real positive coupling constants. Normalization of the coupling constants is chosen such that they reduce to those of the Calogero models in the small coupling limits discussed below. The D model is obtained from the B model by $g_L = g$ and $g_S = 0$. The B and C models are special cases of the $\widetilde{BC}$ model. In contrast to the Calogero case, those based on the B and C systems are different. The difference of the length of the roots is immaterial, since it can be absorbed by the coupling constants normalization. It is rather elementary to derive the forms of the Hamiltonians of the B and C systems from those of the D and A systems by *folding* (see (2.52) and (2.50)). In all these cases the 'potential' P (2.47) grows linearly in $|q|$ as $|q| \to \infty$. Except for the $\widetilde{BC}$ case, there are simple identities: $\sum_j (V_j(q) + V_j^*(q)) = \text{const}$.

In the limit of small coupling constants, namely, by recovering the deformation parameter β',

$$\left(p_j, \frac{1}{a}, g, g_L, g_S, g_0, g_1, g_2\right) \to \beta'\left(p_j, \frac{1}{a}, g, g_L, g_S, g_0, g_1, g_2\right), \tag{2.58}$$

and taking $\beta' \to 0$ limit, the Hamiltonian (2.42) tends to that of the corresponding classical Calogero system (2.8) with $\omega = \frac{1}{a}$ ($\widetilde{BC}_r$ tends to C_r with $g_S = g_0$, $g_L = g_1 + g_2$)

$$\frac{1}{\beta'^2} H(p, q) \to H_{\text{Calogero}}(p, q) + \text{const}. \tag{2.59}$$

It is interesting to note that the equations determining the equilibrium (2.48), in general, can be cast in a form which looks similar to the *Bethe ansatz* equation. For the elementary potential (2.54)–(2.57), the equilibrium positions $\{\bar{q}_j\}$ are determined by

$$A_{r-1}: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g}{\bar{q}_j - \bar{q}_k + \mathrm{i}g} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j}, \tag{2.60}$$

$$B_r: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g_L}{\bar{q}_j - \bar{q}_k + \mathrm{i}g_L} \frac{\bar{q}_j + \bar{q}_k - \mathrm{i}g_L}{\bar{q}_j + \bar{q}_k + \mathrm{i}g_L} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j} \left(\frac{2\bar{q}_j + \mathrm{i}g_S}{2\bar{q}_j - \mathrm{i}g_S}\right)^2, \tag{2.61}$$

$$C_r: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g_S}{\bar{q}_j - \bar{q}_k + \mathrm{i}g_S} \frac{\bar{q}_j + \bar{q}_k - \mathrm{i}g_S}{\bar{q}_j + \bar{q}_k + \mathrm{i}g_S} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j} \frac{\bar{q}_j + \mathrm{i}g_L}{\bar{q}_j - \mathrm{i}g_L}, \tag{2.62}$$

$$\widetilde{BC}_r: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g_0}{\bar{q}_j - \bar{q}_k + \mathrm{i}g_0} \frac{\bar{q}_j + \bar{q}_k - \mathrm{i}g_0}{\bar{q}_j + \bar{q}_k + \mathrm{i}g_0} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j} \frac{\bar{q}_j + \mathrm{i}g_1}{\bar{q}_j - \mathrm{i}g_1} \frac{\bar{q}_j + \mathrm{i}g_2}{\bar{q}_j - \mathrm{i}g_2}, \tag{2.63}$$

$$D_r: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g}{\bar{q}_j - \bar{q}_k + \mathrm{i}g} \frac{\bar{q}_j + \bar{q}_k - \mathrm{i}g}{\bar{q}_j + \bar{q}_k + \mathrm{i}g} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j}. \tag{2.64}$$

In the above small coupling limits, these equations reduce to (2.14)–(2.17). Thus the *Bethe ansatz*-like equations (2.60)–(2.64) would give deformation of the Hermite and Laguerre polynomials, as we will see in sections 3.1 and 4.1. They are not the so-called *q-deformed* Hermite or Laguerre polynomials [15].

Quadratic confining potential case. In this case the elementary potential function v is the same as before, but w is different. For the simply-laced root systems A and D, the elementary potential functions are

$$A, D: \quad v(x) = 1 - \mathrm{i}\frac{g}{x}, \qquad w(x) = \left(1 + \mathrm{i}\frac{x}{a}\right)\left(1 + \mathrm{i}\frac{x}{b}\right) \qquad (a, b, g > 0). \tag{2.65}$$

For the non-simply-laced root systems B, C and $\widetilde{BC}$, we have ($g_L, g_S, g_0, g_1, g_2 > 0$):

$$B: \quad v(x) = 1 - \mathrm{i}\frac{g_L}{x}, \qquad w(x) = \left(1 + \mathrm{i}\frac{x}{a}\right)\left(1 + \mathrm{i}\frac{x}{b}\right)\left(1 - \mathrm{i}\frac{g_S}{2x}\right)^2, \tag{2.66}$$

$$C: \quad v(x) = 1 - \mathrm{i}\frac{g_S}{x}, \qquad w(x) = \left(1 + \mathrm{i}\frac{x}{a}\right)\left(1 + \mathrm{i}\frac{x}{b}\right)\left(1 - \mathrm{i}\frac{g_L}{x}\right), \tag{2.67}$$

$$\widetilde{BC}: \quad v(x) = 1 - \mathrm{i}\frac{g_0}{x}, \qquad w(x) = \left(1 + \mathrm{i}\frac{x}{a}\right)\left(1 + \mathrm{i}\frac{x}{b}\right)\left(1 - \mathrm{i}\frac{g_1}{x}\right)\left(1 - \mathrm{i}\frac{g_2}{x}\right). \tag{2.68}$$

The D model can be considered as a special case of the B model by $g_L = g$ and $g_S = 0$. The B and C models are special cases of $\widetilde{BC}$ model. As in the previous case, the forms of the elementary potential function w for B and C systems are determined from those of the

D and A systems by folding (2.52), (2.50). In all these cases the 'potential' P (2.47) grows quadratically in $|q|$ as $|q| \to \infty$. The small coupling limit (2.58) (and $\frac{1}{b} \to \frac{\beta'}{b}$) gives the same classical Calogero systems as before (2.59), (2.8) with $\omega = \frac{1}{a} + \frac{1}{b}$.

The equations (2.48) determining the equilibrium positions $\{\bar{q}_j\}$ for the elementary potential (2.65)–(2.68) are expressed in a form similar to the *Bethe ansatz* equation:

$$A_{r-1}\colon \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g}{\bar{q}_j - \bar{q}_k + \mathrm{i}g} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j}\frac{b - \mathrm{i}\bar{q}_j}{b + \mathrm{i}\bar{q}_j}, \tag{2.69}$$

$$B_r\colon \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g_L}{\bar{q}_j - \bar{q}_k + \mathrm{i}g_L}\frac{\bar{q}_j + \bar{q}_k - \mathrm{i}g_L}{\bar{q}_j + \bar{q}_k + ig_L} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j}\frac{b - \mathrm{i}\bar{q}_j}{b + \mathrm{i}\bar{q}_j}\left(\frac{2\bar{q}_j + \mathrm{i}g_S}{2\bar{q}_j - \mathrm{i}g_S}\right)^2, \tag{2.70}$$

$$C_r\colon \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g_S}{\bar{q}_j - \bar{q}_k + \mathrm{i}g_S}\frac{\bar{q}_j + \bar{q}_k - \mathrm{i}g_S}{\bar{q}_j + \bar{q}_k + \mathrm{i}g_S} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j}\frac{b - \mathrm{i}\bar{q}_j}{b + \mathrm{i}\bar{q}_j}\frac{\bar{q}_j + \mathrm{i}g_L}{\bar{q}_j - \mathrm{i}g_L}, \tag{2.71}$$

$$\widetilde{BC}_r\colon \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g_0}{\bar{q}_j - \bar{q}_k + \mathrm{i}g_0}\frac{\bar{q}_j + \bar{q}_k - \mathrm{i}g_0}{\bar{q}_j + \bar{q}_k + \mathrm{i}g_0} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j}\frac{b - \mathrm{i}\bar{q}_j}{b + \mathrm{i}\bar{q}_j}\frac{\bar{q}_j + \mathrm{i}g_1}{\bar{q}_j - \mathrm{i}g_1}\frac{\bar{q}_j + \mathrm{i}g_2}{\bar{q}_j - \mathrm{i}g_2}, \tag{2.72}$$

$$D_r\colon \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i}g}{\bar{q}_j - \bar{q}_k + \mathrm{i}g}\frac{\bar{q}_j + \bar{q}_k - \mathrm{i}g}{\bar{q}_j + \bar{q}_k + \mathrm{i}g} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j}\frac{b - \mathrm{i}\bar{q}_j}{b + \mathrm{i}\bar{q}_j}. \tag{2.73}$$

They define another type of deformation of the Hermite and Laguerre polynomials, since the small coupling limit of the above Bethe ansatz-like equations gives the same equations as before (2.14)–(2.17), determining the zeros of the Hermite and Laguerre polynomials, with $\omega = \frac{1}{a} + \frac{1}{b}$. These will be discussed in sections 3.2 and 4.2.

2.2.2. Ruijsenaars–Sutherland systems. The discrete analogue of the Sutherland systems [2], to be called the Ruijsenaars–Sutherland systems, was introduced originally by Ruijsenaars and Schneider [6] for the A type root system. The quantum eigenfunctions of the A type Ruijsenaars–Sutherland systems are called Macdonald polynomials [19], which are a one-parameter deformation (q-deformation) of the Jack polynomials [13]. Here we will discuss the Ruijsenaars–Sutherland systems for all the classical root systems, A, B, C, D and BC [8]. The structure of the functions $\{V_j(q)\}$, (2.43) and (2.44) is the same as in the Ruijsenaars–Calogero systems, but the elementary potential functions v and w are trigonometric instead of rational. Because of the identity $\sum_{j=1}^{r}\{V_j(q) + V_j^*(q)\} = \text{const}$, the Hamiltonian (2.42) could be replaced by a simpler one (2.45).

The elementary potential functions v and w are

$$A, D\colon \quad v(x) = 1 - \mathrm{i}\tanh g \cot x, \qquad w(x) = 1, \tag{2.74}$$

$$B\colon \quad v(x) = 1 - \mathrm{i}\tanh g_L \cot x, \qquad w(x) = \left(1 - \mathrm{i}\tanh \frac{g_S}{2}\cot x\right)^2, \tag{2.75}$$

$$B'\colon \quad v(x) = 1 - \mathrm{i}\tanh g_L \cot x, \qquad w(x) = 1 - \mathrm{i}\tanh g_S \cot x, \tag{2.76}$$

$$C\colon \quad v(x) = 1 - \mathrm{i}\tanh g_S \cot x, \qquad w(x) = 1 - \mathrm{i}\tanh 2g_L \cot 2x, \tag{2.77}$$

$$C'\colon \quad v(x) = 1 - \mathrm{i}\tanh g_S \cot x, \qquad w(x) = (1 - \mathrm{i}\tanh g_L \cot 2x)^2, \tag{2.78}$$

$$B'C: \quad v(x) = 1 - \mathrm{i}\tanh g_M \cot x, \qquad w(x) = (1 - \mathrm{i}\tanh g_S \cot x)(1 - \mathrm{i}\tanh 2g_L \cot 2x), \tag{2.79}$$

with similar coupling constant notation as in the rational cases. Normalization of the coupling constants is chosen such that they reduce to those of the Sutherland models in the small coupling limits discussed below. The D model is the special case of the B model by $g_L = g$ and $g_S = 0$. The B' and C models are special cases of the $B'C$ model. As in the rational cases, the forms of the elementary potential function w for the B and C systems are determined from those of the D and A systems by folding (2.52), (2.50).

The original Sutherland models are obtained in the limit in which all the coupling constant(s) become infinitesimally small. By recovering the deformation parameter β',

$$(p_j, g, g_L, g_M, g_S) \to \beta'(p_j, g, g_L, g_M, g_S), \tag{2.80}$$

and taking $\beta' \to 0$ limit, the Hamiltonian (2.42) tends to that of the corresponding classical Sutherland system (2.8)

$$\frac{1}{\beta'^2} H(p, q) \to H_{\mathrm{Sutherland}}(p, q) + \mathrm{const.} \tag{2.81}$$

In ‘strong’ coupling limits, $g, g_L, g_M, g_S \to +\infty$, the elementary potential functions v and w take simple forms:

$$v(x) \to 1 - \mathrm{i}\cot x, \qquad w(x) \to 1, \qquad 1 - \mathrm{i}\cot x, \qquad (1 - \mathrm{i}\cot x)^2, \quad \text{etc.} \tag{2.82}$$

The deformed polynomials take simple forms in this limit as we will see in section 5.

The equations (2.48) determining the equilibrium positions $\{\bar{q}_j\}$ for the elementary potential (2.74)–(2.79) are expressed in a form similar to the *Bethe ansatz* equation:

$$A_{r-1}: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\tan(\bar{q}_j - \bar{q}_k) - \mathrm{i}\tanh g}{\tan(\bar{q}_j - \bar{q}_k) + \mathrm{i}\tanh g} = 1, \tag{2.83}$$

$$B_r: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\tan(\bar{q}_j - \bar{q}_k) - \mathrm{i}\tanh g_L}{\tan(\bar{q}_j - \bar{q}_k) + \mathrm{i}\tanh g_L} \frac{\tan(\bar{q}_j + \bar{q}_k) - \mathrm{i}\tanh g_L}{\tan(\bar{q}_j + \bar{q}_k) + \mathrm{i}\tanh g_L} = \left(\frac{\tan\bar{q}_j + \mathrm{i}\tanh\frac{g_S}{2}}{\tan\bar{q}_j - \mathrm{i}\tanh\frac{g_S}{2}} \right)^2, \tag{2.84}$$

$$B'_r: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\tan(\bar{q}_j - \bar{q}_k) - \mathrm{i}\tanh g_L}{\tan(\bar{q}_j - \bar{q}_k) + \mathrm{i}\tanh g_L} \frac{\tan(\bar{q}_j + \bar{q}_k) - \mathrm{i}\tanh g_L}{\tan(\bar{q}_j + \bar{q}_k) + \mathrm{i}\tanh g_L} = \frac{\tan\bar{q}_j + \mathrm{i}\tanh g_S}{\tan\bar{q}_j - \mathrm{i}\tanh g_S}, \tag{2.85}$$

$$C_r: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\tan(\bar{q}_j - \bar{q}_k) - \mathrm{i}\tanh g_S}{\tan(\bar{q}_j - \bar{q}_k) + \mathrm{i}\tanh g_S} \frac{\tan(\bar{q}_j + \bar{q}_k) - \mathrm{i}\tanh g_S}{\tan(\bar{q}_j + \bar{q}_k) + \mathrm{i}\tanh g_S} = \frac{\tan 2\bar{q}_j + \mathrm{i}\tanh 2g_L}{\tan 2\bar{q}_j - \mathrm{i}\tanh 2g_L}, \tag{2.86}$$

$$C'_r: \quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\tan(\bar{q}_j - \bar{q}_k) - \mathrm{i}\tanh g_S}{\tan(\bar{q}_j - \bar{q}_k) + \mathrm{i}\tanh g_S} \frac{\tan(\bar{q}_j + \bar{q}_k) - \mathrm{i}\tanh g_S}{\tan(\bar{q}_j + \bar{q}_k) + \mathrm{i}\tanh g_S} = \left(\frac{\tan 2\bar{q}_j + \mathrm{i}\tanh g_L}{\tan 2\bar{q}_j - \mathrm{i}\tanh g_L} \right)^2, \tag{2.87}$$

$$\begin{aligned} B'C_r: \quad & \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\tan(\bar{q}_j - \bar{q}_k) - \mathrm{i}\tanh g_M}{\tan(\bar{q}_j - \bar{q}_k) + \mathrm{i}\tanh g_M} \frac{\tan(\bar{q}_j + \bar{q}_k) - \mathrm{i}\tanh g_M}{\tan(\bar{q}_j + \bar{q}_k) + \mathrm{i}\tanh g_M} \\ & = \frac{\tan\bar{q}_j + \mathrm{i}\tanh g_S}{\tan\bar{q}_j - \mathrm{i}\tanh g_S} \frac{\tan 2\bar{q}_j + \mathrm{i}\tanh 2g_L}{\tan 2\bar{q}_j - \mathrm{i}\tanh 2g_L}, \end{aligned} \tag{2.88}$$

$$D_r\colon\quad \prod_{\substack{k=1\\k\neq j}}^{r} \frac{\tan(\bar q_j-\bar q_k)-\mathrm{i}\tanh g}{\tan(\bar q_j-\bar q_k)+\mathrm{i}\tanh g}\,\frac{\tan(\bar q_j+\bar q_k)-\mathrm{i}\tanh g}{\tan(\bar q_j+\bar q_k)+\mathrm{i}\tanh g}=1. \tag{2.89}$$

From the property mentioned after (2.48) and the fact that (2.83)–(2.89) are the equations of $\{\tan\bar q_j\}$, we can restrict $\bar q_j$ to $0\leqslant\bar q_j\leqslant\pi/2$ (except for the A_{r-1} case). In the small coupling limit, these equations (2.83)–(2.89) tend to (2.26)–(2.30). Thus, the *Bethe ansatz*-like equations (2.83)–(2.89) would give deformation of the Chebyshev and Jacobi polynomials as we will see in section 5.

2.3. Orthogonal polynomials and three-term recurrences

It is well known that orthogonal polynomials satisfy three-term recurrence [18, 20] and conversely a sequence of polynomials satisfying three-term recurrence are orthogonal with respect to certain inner product with some weight function. Here we will introduce appropriate notation by taking the classical orthogonal polynomials as examples.

Let $\{f_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials with real coefficients. That is $f_n(x)$ is a degree n polynomial in x and they are mutually orthogonal $(f_n,f_m)=h_n\delta_{n,m}$ $(h_n>0)$ with respect to an (positive definite) inner product $(f,g)=\int f(x)g(x)\mathbf{w}(x)\,\mathrm{d}x$ ($\mathbf{w}(x)$ is a weight function). Let $f_n^{\text{monic}}(x)$ be a monic one, $f_n(x)=c_nf_n^{\text{monic}}(x)=c_n(x^n+\cdots)$. Then $f_n^{\text{monic}}(x)$ satisfies three-term recurrence:

$$f_{n+1}^{\text{monic}}(x)-(x-a_n)f_n^{\text{monic}}(x)+b_nf_{n-1}^{\text{monic}}(x)=0\qquad(n\geqslant 0), \tag{2.90}$$

where we have set $f_{-1}^{\text{monic}}(x)=0$,[3] and a_n $(n\geqslant 0)$ and b_n $(n\geqslant 1$, b_0 is unnecessary, $b_n>0)$ are real numbers. The constants a_n, b_n and h_n are given by

$$a_n=\frac{(xf_n(x),f_n(x))}{(f_n(x),f_n(x))},\qquad b_n=\frac{c_{n-1}^2}{c_n^2}\frac{(f_n(x),f_n(x))}{(f_{n-1}(x),f_{n-1}(x))},\qquad h_n=(1,1)c_n^2\prod_{j=1}^{n}b_j. \tag{2.91}$$

Namely $f_n(x)$ satisfies the three-term recurrence

$$\frac{c_n}{c_{n+1}}f_{n+1}(x)-(x-a_n)f_n(x)+b_n\frac{c_n}{c_{n-1}}f_{n-1}(x)=0\qquad(n\geqslant 0). \tag{2.92}$$

For $a_n=0$ $(n\geqslant 0)$ case, $f_n(x)$ has a definite parity, $f_n(-x)=(-1)^nf_n(x)$, and the constant term of the even polynomial is

$$f_{2n}(0)=(-1)^nc_{2n}\prod_{j=1}^{n}b_{2j-1}. \tag{2.93}$$

Conversely, if $\{f_n(x)\}$ is defined by the three-term recurrence (2.90), namely, real numbers a_n $(n\geqslant 0)$, b_n $(n\geqslant 1, b_n>0)$ and c_n $(n\geqslant 0, c_n\neq 0)$ are given, then $\{f_n(x)\}$ is a sequence of orthogonal polynomials with respect to some (positive definite) inner product $(\cdot,\cdot)$.

In the rest of this subsection we summarize the three-term recurrence, generating functions, differential equations, etc for the Hermite, Laguerre and Jacobi polynomials for later comparison with the corresponding quantities of the deformed polynomials.

[3] Hereafter we adopt the convention $f_{-1}(x)=0$ and $f_0(x)=1$ for all the polynomials in this paper.

The Hermite polynomials $H_n(x)$ (2.18) are orthogonal with respect to the inner product $(f,g)=\int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x$, $(h_n=2^n n!\sqrt{\pi})$ and satisfy the three-term recurrence (2.90) with

$$a_n=0, \qquad b_n=\frac{n}{2}, \qquad c_n=2^n, \tag{2.94}$$

$$H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x)=0. \tag{2.95}$$

The generating function and orthogonality are

$$G(t,x)\stackrel{\text{def}}{=}\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)=\mathrm{e}^{-t^2+2xt}, \qquad (G(t,x),G(s,x))=\sqrt{\pi}\,\mathrm{e}^{2ts}. \tag{2.96}$$

The Laguerre polynomials $L_n^{(\alpha)}(x)$ (2.19) are orthogonal with respect to the inner product $(f,g)=\int_0^{\infty} f(x)g(x)x^{\alpha}\,\mathrm{e}^{-x}\,\mathrm{d}x$, $(h_n(\alpha)=\Gamma(\alpha+n+1)/n!$, $\mathrm{Re}\,\alpha>-1)$ and satisfy the three-term recurrence (2.90) with

$$a_n=2n+1+\alpha, \qquad b_n=n(n+\alpha), \qquad c_n=(-1)^n/n!, \tag{2.97}$$

$$(n+1)L_{n+1}^{(\alpha)}(x)+(x-(2n+\alpha+1))L_n^{(\alpha)}(x)+(n+\alpha)L_{n-1}^{(\alpha)}(x)=0. \tag{2.98}$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (2.34) are orthogonal with respect to $(f,g)=\int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x$, $\left(h_n(\alpha,\beta)=\frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1}\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\right)$ and satisfy the three-term recurrence (2.90) with

$$a_n=\frac{\beta^2-\alpha^2}{d_{2n}d_{2n+2}}, \qquad b_n=4n(n+\alpha)(n+\beta)\times\frac{d_n}{d_{2n-1}d_{2n}^2d_{2n+1}}, \tag{2.99}$$

$$c_n=2^{-n}\binom{\alpha+\beta+2n}{n}, \qquad d_m=\alpha+\beta+m, \tag{2.100}$$

$$\begin{aligned}&2(n+1)(\alpha+\beta+n+1)(\alpha+\beta+2n)P_{n+1}^{(\alpha,\beta)}(x)\\&\quad-(\alpha+\beta+2n+1)((\alpha+\beta+2n)(\alpha+\beta+2n+2)x+\alpha^2-\beta^2)P_n^{(\alpha,\beta)}(x)\\&\quad+2(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)P_{n-1}^{(\alpha,\beta)}(x)=0.\end{aligned} \tag{2.101}$$

The Gegenbauer polynomial $C_n^{(\alpha+\frac{1}{2})}(x)$ (2.35) is a special case of the Jacobi polynomial $C_n^{(\alpha+\frac{1}{2})}(x)=\binom{2\alpha+n}{n}\binom{\alpha+n}{n}^{-1}P_n^{(\alpha,\alpha)}(x)$. The Chebyshev polynomial of the first kind $T_n(x)$ is also a special case of the Jacobi polynomial $T_n(x)=2^{-1}\binom{2n-1}{n}^{-1}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$. The Legendre polynomial is $P_n(x)=P_n^{(0,0)}(x)$.

The differential equations of the Hermite, Laguerre and Jacobi polynomials are

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}H_n(x)-2x\frac{\mathrm{d}}{\mathrm{d}x}H_n(x)+2nH_n(x)=0, \tag{2.102}$$

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}L_n^{(\alpha)}(x)+(\alpha+1-x)\frac{\mathrm{d}}{\mathrm{d}x}L_n^{(\alpha)}(x)+nL_n^{(\alpha)}(x)=0, \tag{2.103}$$

$$\begin{aligned}&(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_n^{(\alpha,\beta)}(x)+(\beta-\alpha-(\alpha+\beta+2)x)\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x)\\&\quad+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0.\end{aligned} \tag{2.104}$$

3. Deformation of the Hermite polynomial

3.1. Linear confining potential case (one-parameter deformation)

For the solution $\{\bar{q}_j\}$ of the A_{r-1} equation (2.60), let us define

$$\bar{q}_j = \sqrt{ag}\,y_j, \qquad \delta = \frac{g}{a}, \tag{3.1}$$

and introduce a degree r polynomial in x having zeros at $\{y_j\}$:

$$H_r(x,\delta) \stackrel{\text{def}}{=} 2^r \prod_{j=1}^{r}(x - y_j). \tag{3.2}$$

It is a deformation of the Hermite polynomial (2.18) such that

$$\lim_{\delta\to 0} H_r(x,\delta) = H_r(x). \tag{3.3}$$

If $\{\bar{q}_j\}$ is a solution of (2.60), so is $\{-\bar{q}_j\}$, which would imply that the deformed polynomial $H_r(x,\delta)$ has a definite parity

$$H_r(-x,\delta) = (-1)^r H_r(x,\delta), \tag{3.4}$$

as with the original Hermite polynomial $H_r(-x) = (-1)^r H_r(x)$.

The equation for the equilibrium (2.60) can be written as (we replace r by n)

$$\prod_{k=1}^{n} \frac{y_j - y_k - \mathrm{i}\sqrt{\delta}}{y_j - y_k + \mathrm{i}\sqrt{\delta}} = \frac{y_j + \mathrm{i}\frac{1}{\sqrt{\delta}}}{y_j - \mathrm{i}\frac{1}{\sqrt{\delta}}}. \tag{3.5}$$

From this equation, we obtain the following functional equation for $H_n(x,\delta)$ ($n \geqslant 1$):

$$\left(x + \mathrm{i}\frac{1}{\sqrt{\delta}}\right) H_n(x + \mathrm{i}\sqrt{\delta},\delta) - \left(x - \mathrm{i}\frac{1}{\sqrt{\delta}}\right) H_n(x - \mathrm{i}\sqrt{\delta},\delta) = 2\mathrm{i}A_n H_n(x,\delta), \tag{3.6}$$

because the LHS is i times a degree n polynomial in x with real coefficients which vanishes at $x = y_j$. Here $A_n = A_n(\delta)$ is a real constant. This functional equation contains all the information of the equilibrium. The number of unknown coefficients (coefficient of x^k term of $H_n(x,\delta)$ ($k = 0, 1, \ldots, n-1$) and A_n) and the number of equations (coefficient of x^k term of (3.6) ($k = 0, 1, \ldots, n$)) are both n. The constant A_n is determined by the coefficient of x^n term of this equation,

$$A_n = \frac{1}{\sqrt{\delta}}(1 + n\delta). \tag{3.7}$$

The functional equation (3.6) can be written as a difference equation,

$$D^2_{x,\frac{1}{2}\mathrm{i}\sqrt{\delta}} H_n(x,\delta) - 2x D_{x,\mathrm{i}\sqrt{\delta}} H_n(x,\delta) + 2n H_n(x,\delta) = 0, \tag{3.8}$$

where $D_{x,h}$ is a (central) difference operator,

$$D_{x,h} f(x) = \frac{f(x+h) - f(x-h)}{2h}. \tag{3.9}$$

In the $\delta \to 0$ limit, (3.6) reduces to the differential equation of the Hermite polynomial (2.102).

The uniqueness (up to normalization) of the solution of the functional equation (3.6) is easily shown (proposition A.1). Therefore, it is sufficient to construct one solution of (3.6) explicitly. This is done by using the three-term recurrence (see the appendix). The result is as follows; the functional equation (3.6) implies that the deformed Hermite polynomial $H_n(x,\delta)$

satisfies the three-term recurrence (2.90) with

$$a_n = 0, \qquad b_n = \frac{n}{2}\left(1 + \frac{n-1}{2}\delta\right), \qquad c_n = 2^n, \tag{3.10}$$

$$H_{n+1}(x,\delta) - 2xH_n(x,\delta) + (2n + n(n-1)\delta)H_{n-1}(x,\delta) = 0. \tag{3.11}$$

Since δ is positive in this case (3.1), b_n is also positive. Therefore, $H_n(x,\delta)$ is a set of orthogonal polynomials with respect to some positive definite inner product.

Here we present another derivation of this three-term recurrence. Let us consider the generating function

$$G(t,x,\delta) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,\delta), \tag{3.12}$$

which satisfies

$$\left((1+\delta t^2)\frac{\partial}{\partial t} + 2(t-x)\right)G(t,x,\delta) = 0, \tag{3.13}$$

as a consequence of the three-term recursion (3.11). This linear differential equation with the initial condition $G(0,x,\delta) = 1$ can be easily solved and we obtain

$$G(t,x,\delta) = \frac{\exp\left(2x\frac{\arctan\sqrt{\delta}t}{\sqrt{\delta}}\right)}{(1+\delta t^2)^{\frac{1}{\delta}}}. \tag{3.14}$$

In the $\delta \to 0$ limit, this generating function tends to that of the Hermite polynomial (2.96),

$$\lim_{\delta\to 0} G(t,x,\delta) = \mathrm{e}^{-t^2}\mathrm{e}^{2xt} = \mathrm{e}^{-t^2+2xt} = G(t,x). \tag{3.15}$$

The functional equation of $G(t,x,\delta)$ is obtained from (3.6):

$$\left(x + \mathrm{i}\frac{1}{\sqrt{\delta}}\right)G(t, x+\mathrm{i}\sqrt{\delta},\delta) - \left(x - \mathrm{i}\frac{1}{\sqrt{\delta}}\right)G(t,x-\mathrm{i}\sqrt{\delta},\delta) = \frac{2\mathrm{i}}{\sqrt{\delta}}\left(1 + \delta t\frac{\partial}{\partial t}\right)G(t,x,\delta). \tag{3.16}$$

Since the solution of (3.6) is unique (up to normalization), it is sufficient to show that (3.14) satisfies this functional equation. This can be easily verified by explicit calculation, in which the following formula derived from (3.14) is useful,

$$G(t, x \pm \mathrm{i}\sqrt{\delta},\delta) = \frac{1 \pm \mathrm{i}\sqrt{\delta}t}{1 \mp \mathrm{i}\sqrt{\delta}t}G(t,x,\delta). \tag{3.17}$$

The explicit form of the inner product $(f,g) = \int_{-\infty}^{\infty} f(x)g(x)\mathbf{w}(x,\delta)\,\mathrm{d}x$, i.e. the weight function $\mathbf{w}(x,\delta)$ is determined by using the generating function. Here we list main results only without derivation. Let us fix its normalization by $(1,1)_\delta = \sqrt{\pi}$. From the general theory (2.91), the orthogonality of $H_n(x,\delta)$ is

$$(H_n(x,\delta), H_m(x,\delta))_\delta = \delta_{n,m}h_n, \qquad h_n = \sqrt{\pi}2^n n! \prod_{j=0}^{n-1}\left(1 + \frac{1}{2}j\delta\right), \tag{3.18}$$

which leads to

$$(G(t,x,\delta), G(s,x,\delta))_\delta = \sqrt{\pi}(1-\delta ts)^{-\frac{2}{\delta}}. \tag{3.19}$$

Here we have used the identity

$$F(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}\prod_{j=0}^{n-1}\left(1+\frac{1}{2}j\delta\right) = \left(1 - \frac{1}{2}\delta x\right)^{-\frac{2}{\delta}}. \tag{3.20}$$

The weight function is expressed as

$$\mathbf{w}(x,\delta)=\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\sqrt{\pi}}\frac{\cos 2xt}{(\cosh\sqrt{\delta}t)^{\frac{2}{\delta}}}=\frac{2^{\frac{2}{\delta}-1}}{\sqrt{\pi\delta}}B\Big(\frac{1}{\delta}+\mathrm{i}\frac{x}{\sqrt{\delta}},\frac{1}{\delta}-\mathrm{i}\frac{x}{\sqrt{\delta}}\Big) \tag{3.21}$$

$$=\frac{2^{\frac{2}{\delta}-1}}{\sqrt{\pi\delta}}\frac{\Gamma\big(\frac{1}{\delta}+\mathrm{i}\frac{x}{\sqrt{\delta}}\big)\Gamma\big(\frac{1}{\delta}-\mathrm{i}\frac{x}{\sqrt{\delta}}\big)}{\Gamma\big(\frac{2}{\delta}\big)}=\frac{2^{\frac{2}{\delta}-1}}{\sqrt{\pi\delta}}\frac{\big|\Gamma\big(\frac{1}{\delta}+\mathrm{i}\frac{x}{\sqrt{\delta}}\big)\big|^2}{\Gamma\big(\frac{2}{\delta}\big)}. \tag{3.22}$$

The undeformed limit of the weight function $\lim_{\delta\to 0}\mathbf{w}(x,\delta)=\mathrm{e}^{-x^2}$ can be verified by using the asymptotic expansion of the Γ-function. The Taylor series of $\mathbf{w}(x,\delta)$ in powers of δ reads

$$\mathbf{w}(x,\delta)=\mathrm{e}^{-x^2}\bigg(1+\frac{\delta}{24}(3-12x^2+4x^4)+\frac{\delta^2}{5760}(45-1320x^2+2280x^4-864x^6+80x^8)$$
$$+\frac{\delta^3}{2\,903\,040}(-14\,175-71\,820x^2+865\,620x^4-1\,042\,272x^6+386\,928x^8$$
$$-52\,416x^{10}+2240x^{12})+\cdots\bigg). \tag{3.23}$$

Among many interesting properties of $H_n(x,\delta)$, we present only

$$\text{(i)}\quad H_{2n}(0,\delta)=(-1)^n\frac{(2n)!}{n!}\prod_{j=0}^{n-1}(1+j\delta),\qquad \text{see (2.93)}, \tag{3.24}$$

$$\text{(ii)}\quad \frac{\mathrm{d}}{\mathrm{d}x}H_n(x,\delta)=2\sum_{k=0}^{[\frac{n-1}{2}]}(2k)!\binom{n}{2k+1}(-\delta)^k H_{n-1-2k}(x,\delta), \tag{3.25}$$

which is a deformation of $\frac{\mathrm{d}}{\mathrm{d}x}H_n(x)=2nH_n(x)$.

Remark: We may take the three-term recurrence (3.11) as the definition of the deformed Hermite polynomial $H_n(x,\delta)$ for an arbitrary (complex) parameter δ. Then $H_n(x,\delta)$ is a polynomial in x of degree n and in δ of degree $\left[\frac{n}{2}\right]$ with *integer* coefficients. We will not repeat similar remarks which are valid for almost all the deformed polynomials in this paper.

3.2. Quadratic confining potential case (two-parameter deformation)

For the solution $\{\bar{q}_j\}$ of the A_{r-1} equation (2.69), let us define

$$\bar{q}_j=\sqrt{ag}\,y_j,\qquad \delta=\frac{g}{a},\qquad \varepsilon=\frac{a}{b}, \tag{3.26}$$

and introduce a degree r polynomial in x having zeros at $\{y_j\}$:

$$H_r(x,\delta,\varepsilon)\stackrel{\text{def}}{=}2^r\prod_{j=1}^{r}(x-y_j). \tag{3.27}$$

It is a further deformation of the deformed Hermite polynomial defined previously,

$$\lim_{\varepsilon\to 0}H_r(x,\delta,\varepsilon)=H_r(x,\delta),\qquad H_r(-x,\delta,\varepsilon)=(-1)^r H_r(x,\delta,\varepsilon). \tag{3.28}$$

The symmetry between the two parameters $a\leftrightarrow b$ is expressed as

$$H_r(x,\delta\varepsilon,\varepsilon^{-1})=\varepsilon^{\frac{r}{2}}H_r\big(\varepsilon^{-\frac{1}{2}}x,\delta,\varepsilon\big). \tag{3.29}$$

If we define $\hat{H}_r\big(x, \delta_1 \stackrel{\text{def}}{=} \frac{g}{a} = \delta, \delta_2 \stackrel{\text{def}}{=} \frac{g}{b} = \frac{\delta}{\varepsilon}\big) \stackrel{\text{def}}{=} \sqrt{1+\varepsilon}^{\,r} H_r\big(\frac{x}{\sqrt{1+\varepsilon}}, \delta, \varepsilon\big)$, then this symmetry is more manifest, $\hat{H}_r(x, \delta_1, \delta_2) = \hat{H}_r(x, \delta_2, \delta_1)$.

The equation for the equilibrium (2.69) can be written as (we replace r by n)

$$\prod_{k=1}^{n} \frac{y_j - y_k - \mathrm{i}\sqrt{\delta}}{y_j - y_k + \mathrm{i}\sqrt{\delta}} = -\frac{y_j + \mathrm{i}\frac{1}{\sqrt{\delta}}}{y_j - \mathrm{i}\frac{1}{\sqrt{\delta}}} \frac{y_j + \mathrm{i}\frac{1}{\varepsilon\sqrt{\delta}}}{y_j - \mathrm{i}\frac{1}{\varepsilon\sqrt{\delta}}}. \tag{3.30}$$

From this equation, we obtain the following functional equation for $H_n(x, \delta, \varepsilon)$ $(n \geqslant 1)$:

$$\left(x + \mathrm{i}\frac{1}{\sqrt{\delta}}\right)\left(x + \mathrm{i}\frac{1}{\varepsilon\sqrt{\delta}}\right) H_n(x + \mathrm{i}\sqrt{\delta}, \delta, \varepsilon) + \left(x - \mathrm{i}\frac{1}{\sqrt{\delta}}\right)\left(x - \mathrm{i}\frac{1}{\varepsilon\sqrt{\delta}}\right) H_n(x - \mathrm{i}\sqrt{\delta}, \delta, \varepsilon)$$
$$= 2(A_n x^2 + B_n x + C_n) H_n(x, \delta, \varepsilon), \tag{3.31}$$

because the LHS is a degree $n + 2$ polynomial in x with real coefficients which vanishes at $x = y_j$. Here $A_n = A_n(\delta, \varepsilon)$, $B_n = B_n(\delta, \varepsilon)$ and $C_n = C_n(\delta, \varepsilon)$ are real constants:

$$A_n = 1, \qquad B_n = 0, \qquad C_n = -\left(\left(\frac{1}{\delta} + n\right)\varepsilon^{-1} + n + \frac{1}{2}n\,(n-1)\,\delta\right). \tag{3.32}$$

This functional equation contains all the information of the equilibrium. The above functional equation (3.31) reduces to that of $H_n(x, \delta)$ (3.6) in a proper limit $\varepsilon \to 0$. This functional equation can be written as a difference equation,

$$(1 - \delta\varepsilon x^2) D^2_{x,\frac{1}{2}\mathrm{i}\sqrt{\delta}} H_n(x, \delta, \varepsilon) - 2(1+\varepsilon) x D_{x,\mathrm{i}\sqrt{\delta}} H_n(x, \delta, \varepsilon)$$
$$+ 2n\left(1 + \left(1 + \frac{n-1}{2}\delta\right)\varepsilon\right) H_n(x, \delta, \varepsilon) = 0. \tag{3.33}$$

The functional equation (3.31) implies (see the appendix) the three-term recurrence (2.90) for the deformed Hermite polynomial $H_n(x, \delta, \varepsilon)$ with

$$a_n = 0, \qquad b_n = \frac{n}{2}\left(1 + \frac{n-1}{2}\delta\right)\left(1 + \frac{n-1}{2}\delta\varepsilon\right)\frac{d_n}{d_{2n-1}d_{2n+1}}, \qquad c_n = 2^n, \tag{3.34}$$

$$d_m = 1 + \left(1 + \frac{m-2}{2}\delta\right)\varepsilon, \tag{3.35}$$

$$H_{n+1}(x, \delta, \varepsilon) - 2x H_n(x, \delta, \varepsilon) + (2n + n(n-1)\delta)\frac{\big(1 + \frac{n-1}{2}\delta\varepsilon\big)d_n}{d_{2n-1}d_{2n+1}} H_{n-1}(x, \delta, \varepsilon) = 0. \tag{3.36}$$

Since δ and ε are positive in this case (3.26), b_n is also positive. From this three-term recurrence, we obtain the differential equation for the generating function $G(t, x, \delta, \varepsilon) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, \delta, \varepsilon)$,

$$\left((d_{2n-1}d_{2n+1})\Big|_{n \to t\frac{\partial}{\partial t}} \left(\frac{\partial}{\partial t} - 2x\right) + \left(\frac{4}{n} d_{2n-1}d_{2n+1}b_n\right)\Big|_{n \to t\frac{\partial}{\partial t}} t \right) G(t, x, \delta, \varepsilon) = 0, \tag{3.37}$$

which is a third-order linear differential equation with respect to t. The special case $\delta = 0$ gives the original Hermite polynomial, $H_n(x, 0, \varepsilon) = (\sqrt{1+\varepsilon})^{-n} H_n(\sqrt{1+\varepsilon}x)$. The value at the origin of the even polynomial shows a characteristic deformation pattern, see (2.93):

$$H_{2n}(0, \delta, \varepsilon) = (-1)^n \frac{(2n)!}{n!} \prod_{j=0}^{n-1} \frac{(1 + j\delta)(1 + j\delta\varepsilon)}{1 + \big(1 + \big(j + n - \frac{1}{2}\big)\delta\big)\varepsilon}. \tag{3.38}$$

4. Deformation of the Laguerre polynomial

4.1. Linear confining potential case (one- or two-parameter deformation)

$\widetilde{BC}_r$: For the solution $\{\bar{q}_j\}$ of the $\widetilde{BC}_r$ equation (2.63), let us define

$$\bar{q}_j = \sqrt{ag_0}y_j, \qquad \delta = \frac{g_0}{a}, \qquad \alpha = \frac{g_1+g_2}{g_0} - 1, \qquad \gamma = \frac{g_1g_2}{g_0^2}, \tag{4.1}$$

and introduce a degree r polynomial in x, having zeros at $\{y_j^2\}$:

$$L_r^{(\alpha)}(x,\gamma,\delta) \stackrel{\text{def}}{=} \frac{(-1)^r}{r!}\prod_{j=1}^{r}\left(x - y_j^2\right). \tag{4.2}$$

It is a two-parameter deformation of the Laguerre polynomial such that

$$\lim_{\delta\to 0} L_r^{(\alpha)}(x,\gamma,\delta) = L_r^{(\alpha)}(x). \tag{4.3}$$

The equation for the equilibrium (2.63) can be written as (we replace r by n)

$$\prod_{k=1}^{n}\frac{(y_j - \mathrm{i}\sqrt{\delta})^2 - y_k^2}{(y_j + \mathrm{i}\sqrt{\delta})^2 - y_k^2} = \frac{y_j - \mathrm{i}\frac{\sqrt{\delta}}{2}}{y_j + \mathrm{i}\frac{\sqrt{\delta}}{2}}\,\frac{y_j + \mathrm{i}\frac{1}{\sqrt{\delta}}}{y_j - \mathrm{i}\frac{1}{\sqrt{\delta}}}\,\frac{y_j^2 + \mathrm{i}(\alpha+1)\sqrt{\delta}y_j - \gamma\delta}{y_j^2 - \mathrm{i}(\alpha+1)\sqrt{\delta}y_j - \gamma\delta}. \tag{4.4}$$

From this equation, we obtain the following functional equation for $L_n^{(\alpha)}(x,\gamma,\delta)$ $(n \geqslant 1)$:

$$\begin{aligned}
\frac{1}{y}\Bigg(&\left(y - \mathrm{i}\frac{\sqrt{\delta}}{2}\right)\left(y + \mathrm{i}\frac{1}{\sqrt{\delta}}\right)(y^2 + \mathrm{i}(\alpha+1)\sqrt{\delta}y - \gamma\delta)L_n^{(\alpha)}((y+\mathrm{i}\sqrt{\delta})^2,\gamma,\delta)\\
&- \left(y + \mathrm{i}\frac{\sqrt{\delta}}{2}\right)\left(y - \mathrm{i}\frac{1}{\sqrt{\delta}}\right)(y^2 - \mathrm{i}(\alpha+1)\sqrt{\delta}y - \gamma\delta)L_n^{(\alpha)}((y-\mathrm{i}\sqrt{\delta})^2,\gamma,\delta)\Bigg)\\
&= 2\mathrm{i}(A_ny^2 + B_n)L_n^{(\alpha)}(y^2,\gamma,\delta),
\end{aligned} \tag{4.5}$$

because the LHS is i times a degree $2n+2$ even polynomial in y with real coefficients which vanishes at $y = \pm y_j$. Here $A_n = A_n^{(\alpha)}(\gamma,\delta)$ and $B_n = B_n^{(\alpha)}(\gamma,\delta)$ are real constants:

$$A_n = \frac{1}{\sqrt{\delta}}\left(1 + \left(2n + \alpha + \frac{1}{2}\right)\delta\right), \qquad B_n = \frac{\sqrt{\delta}}{2}(\alpha + 1 - 2\gamma + (n+\gamma)\delta). \tag{4.6}$$

This functional equation contains all the information of the equilibrium. The functional equation can be written as a difference equation,

$$\begin{aligned}
&\left(\left(1 + \left(\alpha + \frac{1}{2}\right)\delta\right)y^2 + \left(\frac{\alpha+1}{2} - \gamma\right)\delta + \frac{1}{2}\gamma\delta^2\right)D^2_{y,\frac{1}{2}i\sqrt{\delta}}L_n^{(\alpha)}(y^2,\gamma,\delta)\\
&\quad + (-2y^3 + (2\alpha + 1 + (2\gamma - \alpha - 1)\delta)y + \gamma\delta y^{-1})D_{y,\mathrm{i}\sqrt{\delta}}L_n^{(\alpha)}(y^2,\gamma,\delta)\\
&\quad + (4ny^2 + n\delta)L_n^{(\alpha)}(y^2,\gamma,\delta) = 0.
\end{aligned} \tag{4.7}$$

In the $\delta \to 0$ limit, it becomes,

$$y\frac{\mathrm{d}^2}{\mathrm{d}y^2}L_n^{(\alpha)}(y^2) + (2\alpha + 1 - 2y^2)\frac{\mathrm{d}}{\mathrm{d}y}L_n^{(\alpha)}(y^2) + 4nyL_n^{(\alpha)}(y^2) = 0, \tag{4.8}$$

which is equivalent to the differential equation of the Laguerre polynomial (2.103).

By the same argument as $H_n(x,\delta)$, see the appendix, the functional equation (4.5) implies that the deformed Laguerre polynomial $L_n^{(\alpha)}(x,\gamma,\delta)$ satisfies the three-term recurrence (2.90) with

$$a_n = 2n + \alpha + 1 + (n(2n+1) + 2n\alpha + \gamma)\delta, \tag{4.9}$$

$$b_n = n(n+\alpha)(1+(2n+\alpha-1)\delta+((n-1)(n+\alpha)+\gamma)\delta^2), \qquad c_n = (-1)^n/n!, \tag{4.10}$$

$$(n+1)L_{n+1}^{(\alpha)}(x,\gamma,\delta) + (x-(2n+\alpha+1+(n(2n+1)+2n\alpha+\gamma)\delta))L_n^{(\alpha)}(x,\gamma,\delta) \\ + (n+\alpha)(1+(2n+\alpha-1)\delta+((n-1)(n+\alpha)+\gamma)\delta^2)L_{n-1}^{(\alpha)}(x,\gamma,\delta) = 0. \tag{4.11}$$

In this case (4.1), the parameter ranges are $\delta, \gamma > 0$ and $\alpha > -1$. So b_n is positive. Therefore, $L_n^{(\alpha)}(x,\gamma,\delta)$ is a set of orthogonal polynomials with respect to some positive definite inner product. From this three-term recurrence we obtain the differential equation for the generating function $G^{(\alpha)}(t,x,\gamma,\delta) = \sum_{n=0}^{\infty} t^n L_n^{(\alpha)}(x,\gamma,\delta)$,

$$\left(\frac{\partial}{\partial t} + x - a_n\Big|_{n\to t\frac{\partial}{\partial t}} + \frac{b_n}{n}\Big|_{n\to t\frac{\partial}{\partial t}} t\right) G^{(\alpha)}(t,x,\gamma,\delta) = 0, \tag{4.12}$$

which is a third-order linear differential equation with respect to t.

C_r: For the solution $\{\bar{q}_j\}$ of the C_r equation (2.62), let us define

$$\bar{q}_j = \sqrt{a g_S}\, y_j, \qquad \delta = \frac{g_S}{a}, \qquad \alpha = \frac{g_L}{g_S} - 1, \tag{4.13}$$

and introduce a degree r polynomial in x, having zeros at $\{y_j^2\}$:

$$L_r^{(\alpha)}(x,\delta) \stackrel{\text{def}}{=} \frac{(-1)^r}{r!} \prod_{j=1}^{r} \left(x - y_j^2\right). \tag{4.14}$$

This is a deformation of the Laguerre polynomial such that $\lim_{\delta\to 0} L_r^{(\alpha)}(x,\delta) = L_r^{(\alpha)}(x)$, and obviously it is a special case of $L_n^{(\alpha)}(x,\gamma,\delta)$,

$$L_n^{(\alpha)}(x,\delta) = L_n^{(\alpha)}(x,0,\delta). \tag{4.15}$$

The functional equation for $L_n^{(\alpha)}(x,\delta)$ is easily obtained from that of $L_n^{(\alpha)}(x,\gamma,\delta)$ (4.5) and will not be presented. The three-term recurrence for $L_n^{(\alpha)}(x,\delta)$ reads

$$(n+1)L_{n+1}^{(\alpha)}(x,\delta) + (x-(2n+\alpha+1+n(2n+2\alpha+1)\delta))\, L_n^{(\alpha)}(x,\delta) \\ + (n+\alpha)\,(1+(n-1)\delta)\,(1+(n+\alpha)\delta)\, L_{n-1}^{(\alpha)}(x,\delta) = 0, \tag{4.16}$$

The value at the origin shows a simple deformation pattern

$$L_n^{(\alpha)}(0,\delta) = \binom{n+\alpha}{n} \prod_{j=0}^{n-1} (1+j\delta). \tag{4.17}$$

B_r: For the solution $\{\bar{q}_j\}$ of the B_r equation (2.61), let us define

$$\bar{q}_j = \sqrt{a g_L}\, y_j, \qquad \delta = \frac{g_L}{a}, \qquad \alpha = \frac{g_S}{g_L} - 1, \tag{4.18}$$

and introduce a degree r polynomial in x, having zeros at $\{y_j^2\}$:

$$\tilde{L}_r^{(\alpha)}(x,\delta) \stackrel{\text{def}}{=} \frac{(-1)^r}{r!} \prod_{j=1}^{r} \left(x - y_j^2\right). \tag{4.19}$$

This is a deformation of the Laguerre polynomial such that $\lim_{\delta\to 0} \tilde{L}_r^{(\alpha)}(x,0) = L_r^{(\alpha)}(x)$, and obviously it is a special case of $L_n^{(\alpha)}(x,\gamma,\delta)$

$$\tilde{L}_n^{(\alpha)}(x,\delta) = L_n^{(\alpha)}\left(x, \tfrac{1}{4}(\alpha+1)^2, \delta\right). \tag{4.20}$$

The functional equation of $\tilde{L}_n^{(\alpha)}(x,\delta)$ is easily obtained from that of $L_n^{(\alpha)}(x,\gamma,\delta)$ (4.5) and will not be presented. The three-term recurrence for $\tilde{L}_n^{(\alpha)}(x,\delta)$ reads

$$(n+1)\tilde{L}_{n+1}^{(\alpha)}(x,\delta)+\left(x-\left(2n+\alpha+1+\left(n(2n+2\alpha+1)+\frac{1}{4}(\alpha+1)^2\right)\delta\right)\right)\tilde{L}_n^{(\alpha)}(x,\delta)$$
$$+(n+\alpha)\left(1+\left(n+\frac{\alpha-1}{2}\right)\delta\right)^2\tilde{L}_{n-1}^{(\alpha)}(x,\delta)=0. \qquad (4.21)$$

D_r: As in the Calogero systems, the D_r is a special case $g_S=0$ of the B_r theory described by $\tilde{L}_r^{(-1)}(x,\delta)=L_r^{(-1)}(x,\delta)$, which has a zero at $x=0$ for all r.

4.2. Quadratic confining potential case (two- or three-parameter deformation)

$\widetilde{BC}_r$: For the solution $\{\bar{q}_j\}$ of the $\widetilde{BC}_r$ equation (2.72), let us define

$$\bar{q}_j=\sqrt{ag_0}\,y_j,\qquad \delta=\frac{g_0}{a},\qquad \varepsilon=\frac{b}{a},\qquad \alpha=\frac{g_1+g_2}{g_0}-1,\qquad \gamma=\frac{g_1g_2}{g_0^2}, \qquad (4.22)$$

and introduce a degree r polynomial in x, having zeros at $\{y_j^2\}$:

$$L_r^{(\alpha)}(x,\gamma,\delta,\varepsilon)\stackrel{\text{def}}{=}\frac{(-1)^r}{r!}\prod_{j=1}^{r}\left(x-y_j^2\right). \qquad (4.23)$$

It is a further deformation of the deformed Laguerre polynomial defined previously,

$$\lim_{\varepsilon\to 0}L_r^{(\alpha)}(x,\gamma,\delta,\varepsilon)=L_r^{(\alpha)}(x,\gamma,\delta). \qquad (4.24)$$

The symmetry between the two parameters $a\leftrightarrow b$ is expressed as

$$L_r^{(\alpha)}(x,\gamma,\delta\varepsilon,\varepsilon^{-1})=\varepsilon^r L_r^{(\alpha)}(\varepsilon^{-1}x,\gamma,\delta,\varepsilon). \qquad (4.25)$$

If we define $\hat{L}_r^{(\alpha)}\big(x,\gamma,\delta_1\stackrel{\text{def}}{=}\frac{g}{a}=\delta,\delta_2\stackrel{\text{def}}{=}\frac{g}{b}=\frac{\delta}{\varepsilon}\big)\stackrel{\text{def}}{=}(1+\varepsilon)^r L_r^{(\alpha)}\big(\frac{x}{1+\varepsilon},\gamma,\delta,\varepsilon\big)$, then this symmetry is more manifest, $\hat{L}_r^{(\alpha)}(x,\gamma,\delta_1,\delta_2)=\hat{L}_r^{(\alpha)}(x,\gamma,\delta_2,\delta_1)$.

The equation for the equilibrium (2.72) can be written as (we replace r by n)

$$\prod_{k=1}^{n}\frac{(y_j-\mathrm{i}\sqrt{\delta})^2-y_k^2}{(y_j+\mathrm{i}\sqrt{\delta})^2-y_k^2}=-\frac{y_j-\mathrm{i}\frac{\sqrt{\delta}}{2}}{y_j+\mathrm{i}\frac{\sqrt{\delta}}{2}}\,\frac{y_j+\mathrm{i}\frac{1}{\sqrt{\delta}}}{y_j-\mathrm{i}\frac{1}{\sqrt{\delta}}}\,\frac{y_j+\mathrm{i}\frac{1}{\varepsilon\sqrt{\delta}}}{y_j-\mathrm{i}\frac{1}{\varepsilon\sqrt{\delta}}}\,\frac{y_j^2+\mathrm{i}(\alpha+1)\sqrt{\delta}y_j-\gamma\delta}{y_j^2-\mathrm{i}(\alpha+1)\sqrt{\delta}y_j-\gamma\delta}. \qquad (4.26)$$

The equivalent functional equation for $L_n^{(\alpha)}(x,\gamma,\delta,\varepsilon)$ $(n\geqslant 1)$ reads

$$\frac{1}{y}\Bigg(\left(y-\mathrm{i}\frac{\sqrt{\delta}}{2}\right)\left(y+\mathrm{i}\frac{1}{\sqrt{\delta}}\right)\left(y+\mathrm{i}\frac{1}{\varepsilon\sqrt{\delta}}\right)(y^2+\mathrm{i}(\alpha+1)\sqrt{\delta}y-\gamma\delta)L_n^{(\alpha)}((y+\mathrm{i}\sqrt{\delta})^2,\gamma,\delta,\varepsilon)$$
$$+\left(y+\mathrm{i}\frac{\sqrt{\delta}}{2}\right)\left(y-\mathrm{i}\frac{1}{\sqrt{\delta}}\right)\left(y-\mathrm{i}\frac{1}{\varepsilon\sqrt{\delta}}\right)(y^2-\mathrm{i}(\alpha+1)\sqrt{\delta}y-\gamma\delta)$$
$$\times L_n^{(\alpha)}((y-\mathrm{i}\sqrt{\delta})^2,\gamma,\delta,\varepsilon)\Bigg)$$
$$=2(A_ny^4+B_ny^2+C_n)L_n^{(\alpha)}(y^2,\gamma,\delta,\varepsilon), \qquad (4.27)$$

because the LHS is a degree $2n+4$ even polynomial in y with real coefficients which vanishes at $y=\pm y_j$. Here $A_n=A_n^{(\alpha)}(\gamma,\delta,\varepsilon)$, $B_n=B_n^{(\alpha)}(\gamma,\delta,\varepsilon)$ and $C_n=C_n^{(\alpha)}(\gamma,\delta,\varepsilon)$ are real constants:

$$A_n = 1, \tag{4.28}$$
$$B_n = -\left(\delta^{-1} + 2n + \alpha + \frac{1}{2}\right)\varepsilon^{-1} - \left(2n + \alpha + \frac{1}{2}\right) - \left(\gamma - \frac{\alpha+1}{2} + 2n(n+\alpha)\right)\delta, \tag{4.29}$$
$$C_n = -\frac{1}{2}(\alpha + 1 - 2\gamma + (n+\gamma)\delta)\varepsilon^{-1} - \frac{1}{2}(n+\gamma)\delta - \frac{1}{2}n(n+\alpha)\delta^2. \tag{4.30}$$

This functional equation contains all the information of the equilibrium. In a proper limit $\varepsilon \to 0$, the above functional equation (4.27) reduces to that of $L_n^{(\alpha)}(x,\gamma,\delta)$ (4.5). This functional equation can be written as a difference equation as previous examples.

The functional equation (4.27) implies (see the appendix) that the deformed Laguerre polynomial $L_n^{(\alpha)}(x,\gamma,\delta,\varepsilon)$ satisfies the three-term recurrence (2.90) with,

$$a_n = 2n + \alpha + 1 + \frac{X_0 + X_1\varepsilon + X_2\varepsilon^2}{d_{2n}d_{2n+2}}, \tag{4.31}$$

$$\begin{aligned} b_n = n(n+\alpha)&(1 + (2n+\alpha-1)\delta + ((n-1)(n+\alpha)+\gamma)\delta^2) \\ &\times (1 + (1 + (n-1)\delta)\varepsilon)\,(1 + (2n+\alpha-1)\delta\varepsilon + ((n-1)(n+\alpha)+\gamma)\delta^2\varepsilon^2) \\ &\times \frac{d_n}{d_{2n-1}d_{2n}^2 d_{2n+1}}, \qquad c_n = (-1)^n/n!, \end{aligned} \tag{4.32}$$

$$d_m = 1 + (1 + (m+\alpha-1)\delta)\,\varepsilon. \tag{4.33}$$

Here X_0, X_1 and X_2 are

$$X_0 = (n(2n+1) + 2n\alpha + \gamma)\,\delta, \tag{4.34}$$

$$\begin{aligned} X_1 = &-(2n+\alpha+1) - (2n(n+\alpha+1) + (\alpha+1)^2 - 2\gamma)\delta \\ &+ (n(4n^2 + (6\alpha+1)n + 2\alpha^2 - 1) + (2n+\alpha-1)\gamma)\delta^2, \end{aligned} \tag{4.35}$$

$$\begin{aligned} X_2 = &-(2n+\alpha+1) - (6n^2 + 3(2\alpha+1)n + 2\alpha(\alpha+1) - \gamma)\delta - (4n^3 + 3(2\alpha+1)n^2 \\ &+ (4\alpha^2 + 4\alpha - 1)n + (\alpha-1)(\alpha+1)^2 - (2n+\alpha-1)\gamma)\delta^2 \\ &+ n(n+\alpha)(2n(n+\alpha) - \alpha - 1 + 2\gamma)\delta^3. \end{aligned} \tag{4.36}$$

Namely $L_n^{(\alpha)}(x,\gamma,\delta,\varepsilon)$ satisfies

$$(n+1)L_{n+1}^{(\alpha)}(x,\gamma,\delta,\varepsilon) + (x - a_n)L_n^{(\alpha)}(x,\gamma,\delta,\varepsilon) + \frac{1}{n}b_n L_{n-1}^{(\alpha)}(x,\gamma,\delta,\varepsilon) = 0. \tag{4.37}$$

In this case (4.22), the parameter ranges are $\delta, \varepsilon, \gamma > 0$ and $\alpha > -1$. So b_n is positive. From this three-term recurrence we obtain the differential equation for the generating function $G^{(\alpha)}(t,x,\gamma,\delta,\varepsilon) = \sum_{n=0}^{\infty} t^n L_n^{(\alpha)}(x,\gamma,\delta,\varepsilon)$,

$$\begin{aligned} \Bigg(&\left(d_{2n-1}d_{2n}^2 d_{2n+1}d_{2n+2}\right)\Big|_{n\to t\frac{\partial}{\partial t}} \left(\frac{\partial}{\partial t} + x\right) - \left(d_{2n-1}d_{2n}^2 d_{2n+1}d_{2n+2}a_n\right)\Big|_{n\to t\frac{\partial}{\partial t}} \\ &+ \left(\frac{1}{n}d_{2n-1}d_{2n}^2 d_{2n+1}d_{2n+2}b_n\right)\Bigg|_{n\to t\frac{\partial}{\partial t}} t \Bigg) G^{(\alpha)}(t,x,\gamma,\delta,\varepsilon) = 0, \end{aligned} \tag{4.38}$$

which is an eighth-order linear differential equation with respect to t. The special case of $\delta = 0$ gives the original Laguerre polynomial, $L_n^{(\alpha)}(x,\gamma,0,\varepsilon) = (1+\varepsilon)^{-n}L_n^{(\alpha)}((1+\varepsilon)x)$.

C_r: For the solution $\{\bar{q}_j\}$ of the C_r equation (2.71), let us define

$$\bar{q}_j = \sqrt{ag_S}\,y_j, \qquad \delta = \frac{g_S}{a}, \qquad \varepsilon = \frac{b}{a}, \qquad \alpha = \frac{g_L}{g_S} - 1, \tag{4.39}$$

and introduce a degree r polynomial in x, having zeros at $\{y_j^2\}$:

$$L_r^{(\alpha)}(x,\delta,\varepsilon) \stackrel{\text{def}}{=} \frac{(-1)^r}{r!}\prod_{j=1}^{r}\left(x-y_j^2\right). \tag{4.40}$$

This is a further deformation of the deformed Laguerre polynomial defined previously such that $\lim_{\varepsilon\to 0} L_r^{(\alpha)}(x,\delta,\varepsilon) = L_r^{(\alpha)}(x,\delta)$ (4.14), and obviously it is a special case of $L_r^{(\alpha)}(x,\gamma,\delta,\varepsilon)$,

$$L_r^{(\alpha)}(x,\delta,\varepsilon) = L_r^{(\alpha)}(x,0,\delta,\varepsilon). \tag{4.41}$$

The value at the origin shows a characteristic deformation pattern

$$L_n^{(\alpha)}(0,\delta,\varepsilon) = \binom{n+\alpha}{n}\prod_{j=0}^{n-1}\frac{(1+j\delta)(1+j\delta\varepsilon)}{1+(1+(\alpha+j+r)\delta)\varepsilon}. \tag{4.42}$$

B_r: For the solution $\{\bar{q}_j\}$ of the B_r equation (2.70), let us define

$$\bar{q}_j = \sqrt{ag_L}\,y_j, \qquad \delta = \frac{g_L}{a}, \qquad \varepsilon = \frac{b}{a}, \qquad \alpha = \frac{g_S}{g_L} - 1, \tag{4.43}$$

and introduce a degree r polynomial in x, having zeros at $\{y_j^2\}$:

$$\tilde{L}_r^{(\alpha)}(x,\delta,\varepsilon) \stackrel{\text{def}}{=} \frac{(-1)^r}{r!}\prod_{j=1}^{r}\left(x-y_j^2\right). \tag{4.44}$$

This is a further deformation of the deformed Laguerre polynomial defined previously such that $\lim_{\varepsilon\to 0} \tilde{L}_r^{(\alpha)}(x,\delta,\varepsilon) = \tilde{L}_r^{(\alpha)}(x,\delta)$ (4.19), and obviously it is a special case of $L_r^{(\alpha)}(x,\gamma,\delta,\varepsilon)$,

$$\tilde{L}_r^{(\alpha)}(x,\delta,\varepsilon) = L_r^{(\alpha)}\left(x,\tfrac{1}{4}(\alpha+1)^2,\delta,\varepsilon\right). \tag{4.45}$$

D_r: As in the Calogero systems, the D_r is a special case $g_S = 0$ of the B_r theory described by $\tilde{L}_r^{(-1)}(x,\delta,\varepsilon) = L_r^{(-1)}(x,\delta,\varepsilon)$, which has a zero at $x = 0$ for all r.

Deformation of the identities. Before going to the systems with trigonometric potentials, let us present the one- and two-parameter deformation of the *identities* between the Hermite and Laguerre polynomials, (2.20) and (2.21), which could be considered as consequences of the folding (2.5)–(2.7) and (2.50)–(2.52) of the rational potentials. The one- and two-parameter deformation of the even degree identities (2.20) are

$$2^{-2r}H_{2r}(x,\delta) = (-1)^r r!\,L_r^{(-\frac{1}{2})}(x^2,\delta), \tag{4.46}$$

$$2^{-2r}H_{2r}(x,\delta,\varepsilon) = (-1)^r r!\,L_r^{(-\frac{1}{2})}(x^2,\delta,\varepsilon), \tag{4.47}$$

which are connected with the folding $A_{2r-1} \to C_r$ (2.5). The one- and two-parameter deformation of the odd degree identities (2.21) are

$$2^{-2r-1}H_{2r+1}(x,\delta) = x(-1)^r r!\,L_r^{(\frac{1}{2})}\left(x^2,\tfrac{1}{2},\delta\right), \tag{4.48}$$

$$2^{-2r-1}H_{2r+1}(x,\delta,\varepsilon) = x(-1)^r r!\,L_r^{(\frac{1}{2})}\left(x^2,\tfrac{1}{2},\delta,\varepsilon\right), \tag{4.49}$$

which are related to the folding $A_{2r} \to \widetilde{BC}_r$ (2.7). The one- and two-parameter deformation of the identities between Laguerre polynomials (2.22) are

$$(r+1)\tilde{L}_{r+1}^{(-1)}(x,\delta) = -x\tilde{L}_r^{(1)}(x,\delta), \tag{4.50}$$

$$(r+1)\tilde{L}_{r+1}^{(-1)}(x,\delta,\varepsilon) = -x\tilde{L}_r^{(1)}(x,\delta,\varepsilon), \tag{4.51}$$

which are related to the folding $D_{r+1} \to B_r$ (2.6).

5. Deformation of the Jacobi polynomial

5.1. A_{r-1}

For the A_{r-1} case, the equilibrium position, i.e. the solution $\{\bar{q}_j\}$ of the A_{r-1} equation (2.83), is the same as the original Sutherland system (2.31). Therefore, the polynomial describing the equilibrium is same as the original Sutherland system, the Chebyshev polynomial of the first kind $T_r(x)$ (2.33). In other words, the Chebyshev polynomials are not *deformed* in the present scheme.

5.2. $B'C_r$

$B'C_r$: For the solution $\{\bar{q}_j\}$ of the $B'C_r$ equation (2.88), let us define

$$\delta = \tanh^2 g_M, \qquad \alpha = \frac{\tanh g_S}{\tanh g_M} + \frac{\tanh 2g_L}{2\tanh g_M} - 1, \qquad \beta = \frac{\tanh 2g_L}{2\tanh g_M} - 1, \tag{5.1}$$

and introduce a degree r polynomial in x having zeros at $\{\cos 2\bar{q}_j\}$:

$$P_r^{(\alpha,\beta)}(x,\delta) \stackrel{\text{def}}{=} 2^{-r}\binom{\alpha+\beta+2r}{r}\prod_{j=1}^{r}(x-\cos 2\bar{q}_j). \tag{5.2}$$

It is a deformation of the Jacobi polynomial (2.34) such that

$$\lim_{\delta\to 0} P_r^{(\alpha,\beta)}(x,\delta) = P_r^{(\alpha,\beta)}(x). \tag{5.3}$$

The equation for the equilibrium (2.88) can be written as (we replace r by n)

$$\prod_{k=1}^{n}\frac{\frac{1+\delta}{1-\delta}\cos 2\bar{q}_j + \mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sin 2\bar{q}_j - \cos 2\bar{q}_k}{\frac{1+\delta}{1-\delta}\cos 2\bar{q}_j - \mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sin 2\bar{q}_j - \cos 2\bar{q}_k}$$
$$= \frac{\sqrt{\delta}\cos 2\bar{q}_j + \mathrm{i}\sin 2\bar{q}_j}{\sqrt{\delta}\cos 2\bar{q}_j - \mathrm{i}\sin 2\bar{q}_j}\,\frac{\frac{\sin 2\bar{q}_j}{1+\cos 2\bar{q}_j} + \mathrm{i}(\alpha-\beta)\sqrt{\delta}}{\frac{\sin 2\bar{q}_j}{1+\cos 2\bar{q}_j} - \mathrm{i}(\alpha-\beta)\sqrt{\delta}}\,\frac{\frac{\sin 2\bar{q}_j}{\cos 2\bar{q}_j} + \mathrm{i}2(\beta+1)\sqrt{\delta}}{\frac{\sin 2\bar{q}_j}{\cos 2\bar{q}_j} - \mathrm{i}2(\beta+1)\sqrt{\delta}}. \tag{5.4}$$

Since $\bar{q}_j$ can be restricted to $0 \leqslant \bar{q}_j \leqslant \pi/2$, $\sin 2\bar{q}_j$ is $\sin 2\bar{q}_j = \sqrt{1-\cos^2 2\bar{q}_j}$. From this equation, for $-1 \leqslant x \leqslant 1$, we obtain the following functional equation for $P_n^{(\alpha,\beta)}(x,\delta)$ ($n \geqslant 1$):

$$\begin{aligned}
&\big(\sqrt{\delta}x - \mathrm{i}\sqrt{1-x^2}\big)\big((\alpha-\beta)\sqrt{\delta}(1+x) + \mathrm{i}\sqrt{1-x^2}\big)\big(2(\beta+1)\sqrt{\delta}x + \mathrm{i}\sqrt{1-x^2}\big)\\
&\quad\times P_n^{(\alpha,\beta)}\left(\frac{1+\delta}{1-\delta}x + \mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2},\delta\right) - \big(\sqrt{\delta}x + \mathrm{i}\sqrt{1-x^2}\big)\\
&\quad\times\big((\alpha-\beta)\sqrt{\delta}(1+x) - \mathrm{i}\sqrt{1-x^2}\big)\big(2(\beta+1)\sqrt{\delta}x - \mathrm{i}\sqrt{1-x^2}\big)\\
&\quad\times P_n^{(\alpha,\beta)}\left(\frac{1+\delta}{1-\delta}x - \mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2},\delta\right)\\
&= 2\mathrm{i}\sqrt{1-x^2}(A_n x^2 + B_n x + C_n)P_n^{(\alpha,\beta)}(x,\delta),
\end{aligned}\tag{5.5}$$

because the LHS is $\mathrm{i}\sqrt{1-x^2}$ times a degree $n+2$ polynomial in x with real coefficients which vanishes at $x = \cos 2\bar{q}_j$. Here $A_n = A_n^{(\alpha,\beta)}(\delta)$, $B_n = B_n^{(\alpha,\beta)}(\delta)$ and $C_n = C_n^{(\alpha,\beta)}(\delta)$ are real constants:

$$\begin{aligned}
A_n = -(1-\delta)^{-(n-1)} \times \tfrac{1}{2}&((1+(\alpha-\beta)\sqrt{\delta})(1+2(\beta+1)\sqrt{\delta})(1+\sqrt{\delta})^{2n-1}\\
&+(1-(\alpha-\beta)\sqrt{\delta})(1-2(\beta+1)\sqrt{\delta})(1-\sqrt{\delta})^{2n-1}),
\end{aligned}\tag{5.6}$$

$$B_n = -(\alpha-\beta)(1+2\beta)\delta, \tag{5.7}$$

$$\begin{aligned} C_n = (1-\delta)^{-n} \times \tfrac{1}{2}((1+(\alpha-\beta)\sqrt{\delta})(1+2(\beta+1)\sqrt{\delta})(1+\sqrt{\delta})^{2n-1} \\ +(1-(\alpha-\beta)\sqrt{\delta})(1-2(\beta+1)\sqrt{\delta})(1-\sqrt{\delta})^{2n-1} \\ -2(\alpha-\beta-1)(1+2\beta)\delta(1-\delta)^{n-1}). \end{aligned} \tag{5.8}$$

The functional equation (5.5) contains all the information of the equilibrium. In the $\delta \to 0$ limit, this functional equation reduces to the differential equation of the Jacobi polynomial (2.104).

The three-term recurrence (2.90) for the deformed Jacobi polynomial $P_n^{(\alpha,\beta)}(x,\delta)$ is a consequence of the functional equation (5.5) (see the appendix). The constants in (2.90) are

$$\begin{aligned} a_n = (\beta-\alpha)(1-\delta)^{n-1} \\ \times \frac{1}{2}((1+2\beta)((1+(\alpha-\beta)\sqrt{\delta})(1+2(\beta+1)\sqrt{\delta})(1+\sqrt{\delta})^{2n-1} \\ +(1-(\alpha-\beta)\sqrt{\delta})(1-2(\beta+1)\sqrt{\delta})(1-\sqrt{\delta})^{2n-1}) \\ +2(\alpha-\beta-1)(1-4(\beta+1)^2\delta)(1-\delta)^{n-1}) \times \frac{1}{d_{2n}d_{2n+2}}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} b_n = \frac{1}{2\sqrt{\delta}}((1+\sqrt{\delta})^n-(1-\sqrt{\delta})^n) \\ \times \frac{1}{2}((1+2(\beta+1)\sqrt{\delta})(1+\sqrt{\delta})^{n-1}+(1-2(\beta+1)\sqrt{\delta})(1-\sqrt{\delta})^{n-1}) \\ \times \frac{1}{2\sqrt{\delta}}((1+2(\beta+1)\sqrt{\delta})(1+\sqrt{\delta})^{2n-2}-(1-2(\beta+1)\sqrt{\delta})(1-\sqrt{\delta})^{2n-2}) \\ \times \frac{1}{2}((1+(\alpha-\beta)\sqrt{\delta})(1+\sqrt{\delta})^{n-1}+(1-(\alpha-\beta)\sqrt{\delta})(1-\sqrt{\delta})^{n-1}) \\ \times \frac{1}{2\sqrt{\delta}}((1+(\alpha-\beta)\sqrt{\delta})^2(1+2(\beta+1)\sqrt{\delta})(1+\sqrt{\delta})^{2n-2} \\ -(1-(\alpha-\beta)\sqrt{\delta})^2(1-2(\beta+1)\sqrt{\delta})(1-\sqrt{\delta})^{2n-2}) \\ \times \frac{d_n}{d_{2n-1}d_{2n}^2d_{2n+1}}, \qquad c_n = 2^{-n}\binom{\alpha+\beta+2n}{n}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} d_m = \frac{1}{2\sqrt{\delta}}((1+(\alpha-\beta)\sqrt{\delta})(1+2(\beta+1)\sqrt{\delta})(1+\sqrt{\delta})^{m-2} \\ -(1-(\alpha-\beta)\sqrt{\delta})(1-2(\beta+1)\sqrt{\delta})(1-\sqrt{\delta})^{m-2}). \end{aligned} \tag{5.11}$$

Namely $P_n^{(\alpha,\beta)}(x,\delta)$ satisfies

$$\begin{aligned} \frac{2(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_{n+1}^{(\alpha,\beta)}(x,\delta)+(x-a_n)P_n^{(\alpha,\beta)}(x,\delta) \\ +\frac{(\alpha+\beta+2n-1)(\alpha+\beta+2n)}{2n(\alpha+\beta+n)}b_nP_{n-1}^{(\alpha,\beta)}(x,\delta)=0. \end{aligned} \tag{5.12}$$

In this case (5.1), the parameter ranges are $\delta > 0$ and $\alpha > \beta > -1$. So b_n is positive. From this three-term recurrence we obtain the difference equation for the generating function $G^{(\alpha,\beta)}_{\text{monic}}(t,x,\delta)=\sum_{n=0}^{\infty}t^nP_n^{(\alpha,\beta)\text{monic}}(x,\delta)$,

$$\bigg(\big(d_{2n-1}d_{2n}^2d_{2n+1}d_{2n+2}\big)\Big|_{n\to t\frac{\partial}{\partial t}}\left(\frac{\partial}{\partial t}+x\right)-\big(d_{2n-1}d_{2n}^2d_{2n+1}d_{2n+2}a_n\big)\Big|_{n\to t\frac{\partial}{\partial t}} + \big(d_{2n-1}d_{2n}^2d_{2n+1}d_{2n+2}b_n\big)\Big|_{n\to t\frac{\partial}{\partial t}}t\bigg)G^{(\alpha,\beta)}_{\text{monic}}(t,x,\delta)=0. \tag{5.13}$$

It is interesting to note that the three-term recurrence (5.9), (5.10) simplifies drastically in a 'strong' coupling limit,

$$g_M\to+\infty\Longleftrightarrow\delta\to1,\qquad g_L,g_S:\text{fixed}, \tag{5.14}$$

$$a_0=\frac{\beta-\alpha}{\alpha+\beta+2},\qquad a_1=\frac{(\beta-\alpha)(2\beta+1)}{(\alpha+\beta+2)(\alpha-\beta+1)},\qquad a_n=0,\qquad(n\geqslant2), \tag{5.15}$$

$$b_1=\frac{4(\beta+1)(\alpha+1+(\beta+1)(\alpha-\beta)^2)}{(\alpha-\beta+1)(\alpha+\beta+2)^2(2\beta+3)},\qquad b_2=\frac{\alpha+\beta+2}{2(\alpha-\beta+1)(2\beta+3)},$$
$$b_n=\frac{1}{4},\qquad(n\geqslant3). \tag{5.16}$$

C_r: For the solution $\{\bar q_j\}$ of the C_r equation (2.86), let us define

$$\delta=\tanh^2g_S,\qquad\alpha=\frac{\tanh2g_L}{2\tanh g_S}-1, \tag{5.17}$$

and introduce a degree r polynomial in x having zeros at $\{\cos2\bar q_j\}$:

$$C_r^{(\alpha+\frac12)}(x,\delta)\stackrel{\text{def}}{=}2^r\binom{\alpha-\frac12+r}{r}\prod_{j=1}^{r}(x-\cos2\bar q_j). \tag{5.18}$$

It is a deformation of the Gegenbauer polynomial (2.35) such that

$$\lim_{\delta\to0}C_r^{(\alpha+\frac12)}(x,\delta)=C_r^{(\alpha+\frac12)}(x), \tag{5.19}$$

and obviously it is a special case of $P_r^{(\alpha,\beta)}(x,\delta)$ with definite parity,

$$C_r^{(\alpha+\frac12)}(x,\delta)=\binom{2\alpha+r}{r}\binom{\alpha+r}{r}^{-1}P_r^{(\alpha,\alpha)}(x,\delta),\qquad C_r^{(\alpha+\frac12)}(-x,\delta)=(-1)^rC_r^{(\alpha+\frac12)}(x,\delta). \tag{5.20}$$

The functional equation of $C_n^{(\alpha+\frac12)}(x,\delta)$ is easily obtained from that of $P_n^{(\alpha,\beta)}(x,\delta)$ and will not be presented here. The three-term recurrence for $C_n^{(\alpha+\frac12)}(x,\delta)$ reads

$$\frac{n+1}{2\alpha+2n+1}C_{n+1}^{(\alpha+\frac12)}(x,\delta)+xC_n^{(\alpha+\frac12)}(x,\delta)+\frac{2\alpha+2n-1}{n}b_nC_{n-1}^{(\alpha+\frac12)}(x,\delta)=0. \tag{5.21}$$

B'_r: For the solution $\{\bar q_j\}$ of the B'_r equation (2.85), let us define

$$\delta=\tanh^2g_L,\qquad\alpha=\frac{\tanh g_S}{\tanh g_L}-1, \tag{5.22}$$

and introduce a degree r polynomial in x having zeros at $\{\cos2\bar q_j\}$:

$$\tilde P_r^{(\alpha)}(x,\delta)\stackrel{\text{def}}{=}2^{-r}\binom{\alpha-1+2r}{r}\prod_{j=1}^{r}(x-\cos2\bar q_j). \tag{5.23}$$

Obviously it is a special case of $P_r^{(\alpha,\beta)}(x,\delta)$,

$$\tilde{P}_r^{(\alpha)}(x,\delta) = P_r^{(\alpha,-1)}(x,\delta). \tag{5.24}$$

The functional equation and three-term recurrence of $\tilde{P}_n^{(\alpha)}(x,\delta)$ are obtained from those of $P_n^{(\alpha,\beta)}(x,\delta)$.

5.3. C'_r

C'_r: For the solution $\{\bar{q}_j\}$ of the C'_r equation (2.87), let us define

$$\delta = \tanh^2 g_S, \qquad \alpha = \frac{\tanh g_L}{\tanh g_S} - 1, \tag{5.25}$$

and introduce a degree r polynomial in x having zeros at $\{\cos 2\bar{q}_j\}$:

$$\tilde{C}_r^{(\alpha+\frac{1}{2})}(x,\delta) \stackrel{\text{def}}{=} 2^r \binom{\alpha-\frac{1}{2}+r}{r} \prod_{j=1}^{r} (x - \cos 2\bar{q}_j). \tag{5.26}$$

It is a deformation of the Gegenbauer polynomial (2.35) with definite parity

$$\lim_{\delta\to 0} \tilde{C}_r^{(\alpha+\frac{1}{2})}(x,\delta) = C_r^{(\alpha+\frac{1}{2})}(x), \qquad \tilde{C}_r^{(\alpha+\frac{1}{2})}(-x,\delta) = (-1)^r \tilde{C}_r^{(\alpha+\frac{1}{2})}(x,\delta). \tag{5.27}$$

The functional equation for $\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta)$ $(n \geqslant 1)$ reads

$$\begin{aligned}
&\big(\sqrt{\delta}x - \mathrm{i}\sqrt{1-x^2}\big)\big((\alpha+1)\sqrt{\delta}x + \mathrm{i}\sqrt{1-x^2}\big)^2 \tilde{C}_n^{(\alpha+\frac{1}{2})}\left(\frac{1+\delta}{1-\delta}x + \mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2}, \delta\right)\\
&\qquad - \big(\sqrt{\delta}x + \mathrm{i}\sqrt{1-x^2}\big)\big((\alpha+1)\sqrt{\delta}x - \mathrm{i}\sqrt{1-x^2}\big)^2\\
&\qquad \times \tilde{C}_n^{(\alpha+\frac{1}{2})}\left(\frac{1+\delta}{1-\delta}x - \mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2}, \delta\right)\\
&\quad = 2\mathrm{i}\sqrt{1-x^2}(A_n x^2 + B_n x + C_n)\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta).
\end{aligned} \tag{5.28}$$

Here $A_n = A_n^{(\alpha+\frac{1}{2})}(\delta)$, $B_n = B_n^{(\alpha+\frac{1}{2})}(\delta)$ and $C_n = C_n^{(\alpha+\frac{1}{2})}(\delta)$ are real constants:

$$A_n = -(1-\delta)^{-(n-1)} \times \tfrac{1}{2}((1+(\alpha+1)\sqrt{\delta})^2(1+\sqrt{\delta})^{2n-1} + (1-(\alpha+1)\sqrt{\delta})^2(1-\sqrt{\delta})^{2n-1}), \tag{5.29}$$

$$B_n = 0, \tag{5.30}$$

$$\begin{aligned}
C_n = (1-\delta)^{-n} \times \tfrac{1}{2}&((1+(\alpha+1)\sqrt{\delta})^2(1+\sqrt{\delta})^{2n-1}\\
&+ (1-(\alpha+1)\sqrt{\delta})^2(1-\sqrt{\delta})^{2n-1} - 2\alpha^2\delta(1-\delta)^{n-1}).
\end{aligned} \tag{5.31}$$

This functional equation contains all the information of the equilibrium.

The deformed Gegenbauer polynomial $\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta)$ satisfies the three-term recurrence (2.90) (see the appendix) with

$$a_n = 0, \tag{5.32}$$

$$\begin{aligned}
b_n = \frac{1}{2\sqrt{\delta}}((1+\sqrt{\delta})^n - (1-\sqrt{\delta})^n)\frac{1}{2^2}&((1+(\alpha+1)\sqrt{\delta})(1+\sqrt{\delta})^{n-1}\\
&+ (1-(\alpha+1)\sqrt{\delta})(1-\sqrt{\delta})^{n-1})^2 \frac{d_n}{d_{2n-1}d_{2n+1}},
\end{aligned} \tag{5.33}$$

$$c_n = 2^n \binom{\alpha-\frac{1}{2}+n}{n},$$

$$d_m = \frac{1}{2\sqrt{\delta}}((1+(\alpha+1)\sqrt{\delta})^2(1+\sqrt{\delta})^{m-2} - (1-(\alpha+1)\sqrt{\delta})^2(1-\sqrt{\delta})^{m-2}). \tag{5.34}$$

Namely $\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta)$ satisfies

$$\frac{n+1}{2\alpha+2n+1}\tilde{C}_{n+1}^{(\alpha+\frac{1}{2})}(x,\delta) + x\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta) + \frac{2\alpha+2n-1}{n}b_n\tilde{C}_{n-1}^{(\alpha+\frac{1}{2})}(x,\delta) = 0. \tag{5.35}$$

In this case (5.25), the parameters are $\delta > 0$ and $\alpha > -1$. So b_n is positive. From this three-term recurrence we obtain the difference equation for the generating function $G^{(\alpha+\frac{1}{2})}(t,x,\delta) = \sum_{n=0}^{\infty} t^n \tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta)$ in a similar way to that of $P_n^{(\alpha,\beta)}(x,\delta)$.

In a 'strong' coupling limit

$$g_S \to +\infty \Longleftrightarrow \delta \to 1, \qquad g_L : \text{fixed}, \tag{5.36}$$

the three-term recurrence (5.33), (5.34) simplifies drastically:

$$b_1 = \frac{1}{(\alpha+2)^2}, \qquad b_2 = \frac{\alpha+1}{(\alpha+2)^2}, \qquad b_n = \frac{1}{4}, \quad (n \geqslant 3). \tag{5.37}$$

5.4. B_r

B_r: For the solution $\{\bar{q}_j\}$ of the B_r equation (2.84), let us define

$$\delta = \tanh^2 g_L, \qquad \alpha = \frac{2\tanh\frac{g_S}{2}}{\tanh g_L} - 1, \tag{5.38}$$

and introduce a degree r polynomial in x having zeros at $\{\cos 2\bar{q}_j\}$:

$$\hat{P}_r^{(\alpha)}(x,\delta) \stackrel{\text{def}}{=} 2^{-r}\binom{\alpha-1+2r}{r}\prod_{j=1}^{r}(x-\cos 2\bar{q}_j). \tag{5.39}$$

It is a deformation of the Jacobi polynomial (2.34) such that

$$\lim_{\delta\to 0}\hat{P}_r^{(\alpha)}(x,\delta) = P_r^{(\alpha,-1)}(x). \tag{5.40}$$

The functional equation for $\hat{P}_n^{(\alpha)}(x,\delta)$ $(n \geqslant 1)$ reads

$$\begin{aligned}
&\left(\sqrt{\delta}x - \mathrm{i}\sqrt{1-x^2}\right)\left(\frac{\alpha+1}{2}\sqrt{\delta}(1+x) + \mathrm{i}\sqrt{1-x^2}\right)^2 \hat{P}_n^{(\alpha)}\left(\frac{1+\delta}{1-\delta}x + \mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2},\delta\right)\\
&\quad - \left(\sqrt{\delta}x + \mathrm{i}\sqrt{1-x^2}\right)\left(\frac{\alpha+1}{2}\sqrt{\delta}(1+x) - \mathrm{i}\sqrt{1-x^2}\right)^2\\
&\quad \times \hat{P}_n^{(\alpha)}\left(\frac{1+\delta}{1-\delta}x - \mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2},\delta\right)\\
&= 2\mathrm{i}\sqrt{1-x^2}(A_n x^2 + B_n x + C_n)\hat{P}_n^{(\alpha)}(x,\delta).
\end{aligned} \tag{5.41}$$

Here $A_n = A_n^{(\alpha)}(\delta)$, $B_n = B_n^{(\alpha)}(\delta)$ and $C_n = C_n^{(\alpha)}(\delta)$ are real constants:

$$\begin{aligned}
A_n = -(1-\delta)^{-(n-1)} \times \tfrac{1}{2}\Big(&\big(1+\tfrac{1}{2}(\alpha+1)\sqrt{\delta}\big)^2(1+\sqrt{\delta})^{2n-1}\\
&+\big(1-\tfrac{1}{2}(\alpha+1)\sqrt{\delta}\big)^2(1-\sqrt{\delta})^{2n-1}\Big),
\end{aligned} \tag{5.42}$$

$$B_n = \tfrac{1}{2}(1-\alpha^2)\delta, \tag{5.43}$$

$$C_n = (1-\delta)^{-n} \times \tfrac{1}{2}\big(\big(1+\tfrac{1}{2}(\alpha+1)\sqrt{\delta}\big)^2(1+\sqrt{\delta})^{2n-1} + \big(1-\tfrac{1}{2}(\alpha+1)\sqrt{\delta}\big)^2(1-\sqrt{\delta})^{2n-1} - \big(1+\alpha^2-\tfrac{1}{2}(\alpha+1)^2\delta\big)\delta(1-\delta)^{n-1}\big). \tag{5.44}$$

This functional equation contains all the information of the equilibrium.

The deformed Jacobi polynomial $\hat{P}_n^{(\alpha)}(x,\delta)$ satisfies the three-term recurrence (2.90) (see the appendix) with

$$a_n = (1-\alpha^2)(1-\delta)^{n-1}\frac{d'_n d'_{n+1}}{d_{2n}d_{2n+2}}, \tag{5.45}$$

$$b_n = 4\frac{1}{2\sqrt{\delta}}((1+\sqrt{\delta})^{n-1} - (1-\sqrt{\delta})^{n-1}) \times \frac{1}{2\sqrt{\delta}}((1+\sqrt{\delta})^n - (1-\sqrt{\delta})^n)\frac{{d'_n}^4 d_n d_{n+1}}{d_{2n-1}d_{2n}^2 d_{2n+1}},$$

$$c_n = 2^{-n}\binom{\alpha-1+2n}{n}, \tag{5.46}$$

$$d_m = \frac{1}{2\sqrt{\delta}}\left(\left(1+\frac{1}{2}(\alpha+1)\sqrt{\delta}\right)^2(1+\sqrt{\delta})^{m-2} - \left(1-\frac{1}{2}(\alpha+1)\sqrt{\delta}\right)^2(1-\sqrt{\delta})^{m-2}\right), \tag{5.47}$$

$$d'_m = \frac{1}{2}\left(\left(1+\frac{1}{2}(\alpha+1)\sqrt{\delta}\right)(1+\sqrt{\delta})^{m-1} + \left(1-\frac{1}{2}(\alpha+1)\sqrt{\delta}\right)(1-\sqrt{\delta})^{m-1}\right). \tag{5.48}$$

Namely $\hat{P}_n^{(\alpha)}(x,\delta)$ satisfies

$$\frac{2(n+1)(\alpha+n)}{(\alpha+2n)(\alpha+2n+1)}\hat{P}_{n+1}^{(\alpha)}(x,\delta) + (x-a_n)\hat{P}_n^{(\alpha)}(x,\delta) + \frac{(\alpha+2n-2)(\alpha+2n-1)}{2n(\alpha+n-1)}b_n\hat{P}_{n-1}^{(\alpha)}(x,\delta) = 0, \tag{5.49}$$

In this case (5.38), the parameters are $\delta > 0$ and $\alpha > -1$. So b_n is positive. From this three-term recurrence we obtain the difference equation for the generating function $G^{(\alpha)}(t,x,\delta) = \sum_{n=0}^{\infty} t^n \hat{P}_n^{(\alpha)}(x,\delta)$ in a similar way to that of $P_n^{(\alpha,\beta)}(x,\delta)$.

In a 'strong' coupling limit

$$g_L \to +\infty \Longleftrightarrow \delta \to 1, \qquad g_S : \text{fixed}, \tag{5.50}$$

the three-term recurrence (5.45)–(5.48) simplifies drastically:

$$a_0 = -1, \qquad a_1 = \frac{1-\alpha}{\alpha+3}, \qquad a_n = 0, \qquad (n \geqslant 2), \tag{5.51}$$

$$b_1 = 0, \qquad b_2 = \frac{2(\alpha+1)}{(\alpha+3)^2}, \qquad b_n = \frac{1}{4}, \qquad (n \geqslant 3). \tag{5.52}$$

D_r: As in the Sutherland systems, the D_r is a special case $g_S = 0$ of the B_r theory described by $\hat{P}_r^{(-1)}(x,\delta)$, which has a zero at $x = \pm 1$ for $r \geqslant 2$.

Deformation of the identities. Before closing this section let us briefly discuss the deformation of the identities between the Chebyshev and Jacobi polynomials (2.37), (2.38) and those between the Jacobi polynomials (2.39). As remarked in section 5.1, the Chebyshev polynomials describing the equilibrium of the A_{r-1} systems are not *deformed* in our scheme. Therefore, no deformation of the identities (2.37), (2.38) exists. The folding $D_{r+1} \to B_r$ (2.52) leads to the identity between the deformed Jacobi polynomials $\hat{P}_r^{(\alpha)}(x,\delta)$ (5.39) associated with the B_r systems

$$2(r+1)\hat{P}_{r+1}^{(-1)}(x,\delta) = r(x-1)\hat{P}_r^{(1)}(x,\delta), \tag{5.53}$$

which is a deformation of the identity (2.39). As remarked at the end of section 2.1, we have not been able to deform the identities between the Gegenbauer and Jacobi polynomials (2.40), (2.41) as they do not seem to have a root theoretic explanation.

Among the deformed Jacobi polynomials $P_n^{(\alpha,\beta)}(x,\delta)$ for various α and β, two cases (i) $\alpha=\beta=-1/2$, (ii) $\alpha=-\beta=1/2$ are not *deformed*:

$$P_n^{(-1/2,-1/2)}(x,\delta) = P_n^{(-1/2,-1/2)}(x), \qquad P_n^{(1/2,-1/2)}(x,\delta) = P_n^{(1/2,-1/2)}(x). \tag{5.54}$$

The first is the Chebyshev polynomial of the first kind $T_n(x) \propto \cos n\varphi$, $x=\cos\varphi$, as remarked in section 5.1. The second case is $P_n^{(1/2,-1/2)}(x) \propto \sin((2n+1)\varphi/2)/\sin(\varphi/2)$, $x=\cos\varphi$. In both cases, the zeros of $P_n^{(\alpha,\beta)}(x)$ are *equally spaced*. There is a third case [4, 18] of equally spaced zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, for $\alpha=\beta=1/2$, corresponding to the Chebyshev polynomial of the second kind $U_n(x) \propto \sin n\varphi/\sin\varphi$, $x=\cos\varphi$, which is, interestingly, *deformed*. We have no explanation to offer.

6. Summary and comments

We have derived certain deformation of the classical orthogonal polynomials (the Hermite, Laguerre, Gegenbauer and Jacobi) describing the equilibrium of a class of multi-particle dynamics, the Ruijsenaars–Schneider systems. The R–S systems are 'good' deformation of the Calogero and Sutherland systems whose equilibrium points are described by the zeros of the above classical orthogonal polynomials. As remarked in the text these deformed polynomials do not belong to the q-deformed orthogonal polynomials [15] or their analogues [21].

The quality and quantity of the knowledge of these new polynomials are rather varied. The one-parameter deformation of the Hermite polynomials, section 3.1, is best understood. Its three-term recurrence (3.11) tells that it is the simplest possible deformation which reduces to the original Hermite polynomial without rescaling etc in the zero deformation limit ($\delta \to 0$). As shown in some detail in section 3.1, the generating function (3.14) and the weight function (3.21) are known explicitly. Some identities connecting the Hermite and Laguerre polynomials (2.20), (2.21) are nicely deformed (4.46)–(4.47), (4.48)–(4.49). It is interesting to note that some non-trivial identities of the Hermite polynomials are preserved after deformation. For example, the 'addition theorem' reads

$$\sum_{\substack{n_1,\ldots,n_m=0\\ n_1+\cdots+n_m=n}}^{\infty} \frac{\alpha_1^{n_1}\cdots\alpha_m^{n_m}}{n_1!\cdots n_m!} H_{n_1}(x_1)\cdots H_{n_m}(x_m) = \frac{|\vec{\alpha}|^n}{n!} H_n\left(\frac{\vec{\alpha}\cdot\vec{x}}{|\vec{\alpha}|}\right), \tag{6.1}$$

in which the following notation is used: $\vec{\alpha} = {}^t(\alpha_1,\ldots,\alpha_m)$, $\vec{x} = {}^t(x_1,\ldots,x_m)$, $|\vec{\alpha}| = \sqrt{\alpha_1^2+\cdots+\alpha_m^2}$, $\vec{\alpha}\cdot\vec{x} = \alpha_1 x_1+\cdots+\alpha_m x_m$. The deformed version is

$$\sum_{\substack{n_1,\ldots,n_m=0\\ n_1+\cdots+n_m=n}}^{\infty} \frac{\alpha_1^{n_1}\cdots\alpha_m^{n_m}}{n_1!\cdots n_m!} H_{n_1}\left(x_1,\frac{\delta}{\alpha_1^2}\right)\cdots H_{n_m}\left(x_m,\frac{\delta}{\alpha_m^2}\right) = \frac{|\vec{\alpha}|^n}{n!} H_n\left(\frac{\vec{\alpha}\cdot\vec{x}}{|\vec{\alpha}|},\frac{\delta}{|\vec{\alpha}|^2}\right). \tag{6.2}$$

Both can be derived from the generating functions.

After a long and laborious search through existing literature, we find out that all the deformed orthogonal polynomials introduced in the main text can be related to particular members of the Askey-scheme of hypergeometric orthogonal polynomials [16].

The deformed Hermite polynomial $H_n(x,\delta)$ is related to the Meixner–Pollaczek polynomial $P_n^{(\lambda)}(x;\phi)$ (section 1.7 of [16]):

$$H_n(x,\delta) = n!\sqrt{\delta}^n P_n^{(\frac{1}{\delta})}\left(\frac{x}{\sqrt{\delta}};\frac{\pi}{2}\right). \tag{6.3}$$

This identification allows a simple expression of its general term in terms of a (truncated) hypergeometric series ${}_2F_1$:

$$\begin{aligned} H_n(x,\delta) &= \mathrm{i}^n\sqrt{\delta}^n\left(\frac{\delta}{2}\right)_n {}_2F_1\left(\left.\begin{matrix}-n,\ \frac{1}{\delta}+\mathrm{i}\frac{x}{\sqrt{\delta}}\\ \frac{2}{\delta}\end{matrix}\,\right|\,2\right)\\ &= 2^n i^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\prod_{j=0}^{k-1}\left(\mathrm{i}x+\frac{1}{\sqrt{\delta}}+\sqrt{\delta}j\right)\times\prod_{j=k}^{n-1}\left(\frac{1}{\sqrt{\delta}}+\frac{\sqrt{\delta}}{2}j\right), \end{aligned} \tag{6.4}$$

which is deformation of (2.18). The two-parameter deformation of the Hermite polynomial $H_n(x,\delta,\varepsilon)$ (3.27) is a special case of the continuous Hahn polynomial (section 1.4 of [16]):

$$H_n(x,\delta,\varepsilon) = \frac{2^n n!\sqrt{\delta}^n}{\left(n-1+\frac{2}{\delta}+\frac{2}{\delta\varepsilon}\right)_n} p_n\left(\frac{x}{\sqrt{\delta}};\frac{1}{\delta},\frac{1}{\delta\varepsilon},\frac{1}{\delta},\frac{1}{\delta\varepsilon}\right). \tag{6.5}$$

The (two-parameter) deformed Laguerre polynomial $L_n^{(\alpha)}(x,\gamma,\delta)$ (4.2) is the continuous dual Hahn polynomial (section 1.3 of [16]) with rescaling:

$$L_n^{(\alpha)}(y^2,\gamma,\delta) = \frac{\delta^n}{n!}S_n\left(\frac{y^2}{\delta};\frac{1}{\delta},\alpha_1+1,\alpha_2+1\right), \tag{6.6}$$

in which α_1 and α_2 are the two roots of $x^2-(\alpha-1)x+\gamma-\alpha=0$. The (three-parameter) deformed Laguerre polynomial $L_n^{(\alpha)}(x,\gamma,\delta,\varepsilon)$ (4.23) is the Wilson polynomial (section 1.1 of [16]) with rescaling:

$$L_n^{(\alpha)}(y^2,\gamma,\delta,\varepsilon) = \frac{\delta^n}{n!\left(n+\alpha+\frac{1}{\delta}+\frac{1}{\delta\varepsilon}\right)_n} W_n\left(\frac{y^2}{\delta};\frac{1}{\delta},\frac{1}{\delta\varepsilon},\alpha_1+1,\alpha_2+1\right). \tag{6.7}$$

The deformed Jacobi polynomial $P_n^{(\alpha,\beta)}(x,\delta)$ (5.2) is a special case of the Askey–Wilson polynomial (section 3.1 of [16]):

$$P_n^{(\alpha,\beta)}(x,\delta) = 2^{-2n}\binom{\alpha+\beta+2n}{n}(ab^2q^{n-1};q)_n^{-1}p_n(x;a,b,-b,-1|q), \tag{6.8}$$

$$\begin{gathered} q = \frac{1-\sqrt{\delta}}{1+\sqrt{\delta}} = \mathrm{e}^{-2g_M}, \qquad a = \frac{1-(\alpha-\beta)\sqrt{\delta}}{1+(\alpha-\beta)\sqrt{\delta}} = \mathrm{e}^{-2g_S},\\ b^2 = \frac{1-2(\beta+1)\sqrt{\delta}}{1+2(\beta+1)\sqrt{\delta}} = \mathrm{e}^{-4g_L}. \end{gathered} \tag{6.9}$$

The deformed Gegenbauer polynomial $\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta)$ (5.26) is again a special case of the Askey–Wilson polynomial (section 3.1 of [16]):

$$\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta) = \binom{\alpha-\frac{1}{2}+n}{n}(a^4q^{n-1};q)_n^{-1}p_n(x;a,a,-a,-a|q), \tag{6.10}$$

$$q = \frac{1-\sqrt{\delta}}{1+\sqrt{\delta}} = \mathrm{e}^{-2g_S}, \qquad a^2 = \frac{1-(\alpha+1)\sqrt{\delta}}{1+(\alpha+1)\sqrt{\delta}} = \mathrm{e}^{-2g_L}. \tag{6.11}$$

Another deformation of the Jacobi polynomial $\hat{P}_n^{(\alpha)}(x,\delta)$ (5.39) is also a special case of the Askey–Wilson polynomial (section 3.1 of [16]):

$$\hat{P}_n^{(\alpha)}(x,\delta) = 2^{-2n}\binom{\alpha-1+2n}{n}(a^2q^{n-1};q)_n^{-1}p_n(x;a,a,-1,-1|q), \quad (6.12)$$

$$q = \frac{1-\sqrt{\delta}}{1+\sqrt{\delta}} = \mathrm{e}^{-2g_L}, \qquad a = \frac{1-\frac{1}{2}(\alpha+1)\sqrt{\delta}}{1+\frac{1}{2}(\alpha+1)\sqrt{\delta}} = \mathrm{e}^{-g_S}. \quad (6.13)$$

In all these formulae the Pochhammer symbol $(a)_k = \prod_{j=0}^{k-1}(a+j)$ and its q-extension $(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j)$ are used.

Various 'strong' coupling limits (5.14), (5.36) and (5.50) of the deformed Jacobi type polynomials $P_n^{(\alpha,\beta)}(x,\delta)$, $\tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta)$ and $\hat{P}_n^{(\alpha)}(x,\delta)$ simply correspond to the '*crystal*' limit $q \to 0+$ of the Askey–Wilson polynomials (6.8), (6.10) and (6.12).

For all these polynomials discussed in the present paper, the general term can be expressed in terms of various hypergeometric functions ${}_2F_1$, ${}_3F_2$, ${}_4F_3$ and ${}_4\phi_3$. A Rodrigue type formula, the generating function and the weight function etc can be written by using the general formulae of the Askey-scheme of hypergeometric orthogonal polynomials [16].

As remarked repeatedly in the text, the equations determining the equilibrium positions, (2.60)–(2.64), (2.69)–(2.73) and (2.83)–(2.89) look similar to the Bethe ansatz equation. For the simplest spin 1/2 XXX chain with N sites, the Bethe ansatz equation reads

$$\prod_{\substack{k=1\\k\neq j}}^{l} \frac{\bar{q}_j - \bar{q}_k + 2\mathrm{i}}{\bar{q}_j - \bar{q}_k - 2\mathrm{i}} = \left(\frac{\bar{q}_j + \mathrm{i}}{\bar{q}_j - \mathrm{i}}\right)^N \qquad (j = 1,\ldots,l), \quad (6.14)$$

in which $l \leqslant N$ is the number of up (down) spins. This looks similar to the rational A type equations (2.60) and (2.69) with a special choice of the potential $w(x)$ function, the Nth power rather than linear or quadratic. The corresponding functional relation, Baxter's t-Q relation, looks very much like the functional equations (3.6) and (3.31). It would be interesting to pursue the analogy further.

Acknowledgments

We thank M Rossi and K Hikami for useful discussion. SO and RS are supported in part by grant-in-aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, no. 13135205 and no. 14540259, respectively and SO is also supported by 'Gakubucho Sairyou Keihi' of Faculty of Science, Shinshu University.

Appendix: Relation between the functional equation and the three-term recurrence

In this appendix, we show the relation between the functional equation and the three-term recurrence, without proof. Since the normalization of polynomials is irrelevant, we consider monic polynomials $f_n(x)$ with real coefficients and without superscript 'monic'.

The three-term recurrence of $f_n(x)$ (2.90) is

$$f_{n+1}(x) = (x-a_n)f_n(x) - b_n f_{n-1}(x) \qquad (n \geqslant 0), \quad (\mathrm{A}.1)$$

with $f_{-1}(x) = 0$ and $f_0(x) = 1$. The explicit forms of a_n and b_n can be read from (3.10), (3.34), (4.9)–(4.10), (4.31)–(4.32), (5.9)–(5.10), (5.32)–(5.33), (5.45)–(5.46).

The functional equations for the deformed Hermite, Laguerre and Jacobi polynomials have the following forms ($n \geqslant -1$):

$$\text{Hermite:}\quad h(x)f_n(x+\mathrm{i}\sqrt{\delta})+\epsilon h(x)^* f_n(x-\mathrm{i}\sqrt{\delta}) = 2\mathrm{i}^{\frac{1-\epsilon}{2}} g_n(x) f_n(x), \tag{A.2}$$

$$\text{Laguerre:}\quad h(y)f_n((y+\mathrm{i}\sqrt{\delta})^2)+\epsilon h(y)^* f_n((y-\mathrm{i}\sqrt{\delta})^2) = 2\mathrm{i}^{\frac{1-\epsilon}{2}} g_n(y) f_n(y^2), \tag{A.3}$$

$$\begin{aligned}\text{Jacobi:}\quad & h(x)f_n\left(\frac{1+\delta}{1-\delta}x+\mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2}\right)+\epsilon h(x)^* f_n\left(\frac{1+\delta}{1-\delta}x-\mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2}\right)\\ &= 2\big(\mathrm{i}\sqrt{1-x^2}\big)^{\frac{1-\epsilon}{2}} g_n(x) f_n(x),\end{aligned} \tag{A.4}$$

where ϵ is

$\epsilon = -1$: $H_n(x,\delta), L_n^{(\alpha)}(x,\gamma,\delta), L_n^{(\alpha)}(x,\delta), \tilde{L}_n^{(\alpha)}(x,\delta), P_n^{(\alpha,\beta)}(x,\delta), \tilde{C}_n^{(\alpha+\frac{1}{2})}(x,\delta), \hat{P}_n^{(\alpha)}(x,\delta)$,

$\epsilon = 1$: $H_n(x,\delta,\varepsilon), L_n^{(\alpha)}(x,\gamma,\delta,\varepsilon), L_n^{(\alpha)}(x,\delta,\varepsilon), \tilde{L}_n^{(\alpha)}(x,\delta,\varepsilon), C_n^{(\alpha+\frac{1}{2})}(x,\delta), \tilde{P}_n^{(\alpha)}(x,\delta)$.

The explicit forms of $h(x)$ and $g_n(x)$ can be read from (3.6), (3.31), (4.5), (4.27), (5.5), (5.28) and (5.41)[4].

As the first step, we show the following property of the functional equations:

Proposition A.1. *The solution of the functional equations (A.2)–(A.4), if exists, is unique up to an overall normalization.*

By this proposition, it is sufficient to construct one solution of each functional equation explicitly. We will do this by using the three-term recurrence.

As the second step, we show the following relation:

Proposition A.2. *Three-term recurrence (A.1) implies the relation* ($n \geqslant 0$)[5]

$$\begin{aligned}\text{Hermite:}\quad & \mathrm{i}^{\frac{1+\epsilon}{2}}\sqrt{\delta}h(x)f_n(x+\mathrm{i}\sqrt{\delta})\\ &= ((x-a_n)(g_{n+1}(x)-g_n(x))+\mathrm{i}\sqrt{\delta}g_n(x))f_n(x)\\ &\quad - b_n\,(g_{n+1}(x)-g_{n-1}(x))\,f_{n-1}(x),\end{aligned} \tag{A.5}$$

$$\begin{aligned}\text{Laguerre:}\quad & 2\mathrm{i}^{\frac{1+\epsilon}{2}}\sqrt{\delta}yh(y)f_n((y+\mathrm{i}\sqrt{\delta})^2)\\ &= ((y^2-a_n)(g_{n+1}(y)-g_n(y))+\delta g_n(y)+2\mathrm{i}\sqrt{\delta}yg_n(y))f_n(y^2)\\ &\quad - b_n(g_{n+1}(y)-g_{n-1}(y))f_{n-1}(y^2),\end{aligned} \tag{A.6}$$

$$\begin{aligned}\text{Jacobi:}\quad & \big(\mathrm{i}\sqrt{1-x^2}\big)^{\frac{1+\epsilon}{2}}\frac{2\sqrt{\delta}}{1-\delta}h(x)f_n\left(\frac{1+\delta}{1-\delta}x+\mathrm{i}\frac{2\sqrt{\delta}}{1-\delta}\sqrt{1-x^2}\right)\\ &= \left((x-a_n)g_{n+1}(x)-\left(\frac{1+\delta}{1-\delta}x-a_n\right)g_n(x)+2\mathrm{i}\sqrt{1-x^2}\frac{\sqrt{\delta}}{1-\delta}g_n(x)\right)f_n(x)\\ &\quad - b_n(g_{n+1}(x)-g_{n-1}(x))f_{n-1}(x).\end{aligned} \tag{A.7}$$

[4] The explicit forms of the function $h(x)$ are derived from the equations for the equilibrium. For the deformed Hermite polynomials, $g_n(x)$ are determined by the consistency of this functional equation. For the deformed Laguerre and Jacobi polynomials, however, $g_n(x)$ are not determined uniquely by these functional equations. We have fixed $g_n(x)$ by using some empirical knowledge of their lower degree members.

[5] Although a_n vanishes for the deformed Hermite cases, we keep a_n in this generic formula.

In the proof of this proposition by induction, we encounter the equation,

$$X_n(x) f_n(x) - Y_n(x) b_n f_{n-1}(x) \stackrel{?}{=} 0 \qquad \text{or} \qquad X_n(y) f_n(y^2) - Y_n(y) b_n f_{n-1}(y^2) \stackrel{?}{=} 0. \tag{A.8}$$

Here $X_n(x)$ and $Y_n(x)$ are as follows:
Hermite:

$$\begin{aligned} X_n(x) &= (x-a_n)^2(g_{n+1}(x)-g_n(x)) - (x-a_n)(x-a_{n+1})(g_{n+2}(x)-g_{n+1}(x)) - \delta g_n(x) \\ &\quad - b_n(g_n(x)-g_{n-2}(x)) + b_{n+1}(g_{n+2}(x)-g_n(x)), \end{aligned} \tag{A.9}$$

$$\begin{aligned} Y_n(x) &= (x-a_n)(g_{n+1}(x)-g_{n-1}(x)) - (x-a_{n-1})(g_{n-1}(x)-g_{n-2}(x)) \\ &\quad - (x-a_{n+1})(g_{n+2}(x)-g_{n+1}(x)), \end{aligned} \tag{A.10}$$

Laguerre:

$$\begin{aligned} X_n(y) &= (y^2-a_n-\delta)((y^2-a_n)(g_{n+1}(y)-g_n(y)) + \delta g_n(y)) \\ &\quad - (y^2-a_n)((y^2-a_{n+1})(g_{n+2}(y)-g_{n+1}(y)) + \delta g_{n+1}(y)) - 4\delta y^2 g_n(y) \\ &\quad - b_n(g_n(y)-g_{n-2}(y)) + b_{n+1}(g_{n+2}(y)-g_n(y)), \end{aligned} \tag{A.11}$$

$$\begin{aligned} Y_n(y) &= (y^2-a_n-2\delta)(g_{n+1}(y)-g_{n-1}(y)) - (y^2-a_{n-1})(g_{n-1}(y)-g_{n-2}(y)) \\ &\quad - (y^2-a_{n+1})(g_{n+2}(y)-g_{n+1}(y)), \end{aligned} \tag{A.12}$$

Jacobi:

$$\begin{aligned} X_n(x) &= \left(\frac{1+\delta}{1-\delta}x - a_n\right)\left((x-a_n)g_{n+1}(x) - \left(\frac{1+\delta}{1-\delta}x - a_n\right)g_n(x)\right) \\ &\quad - (x-a_n)\left((x-a_{n+1})g_{n+2}(x) - \left(\frac{1+\delta}{1-\delta}x - a_{n+1}\right)g_{n+1}(x)\right) \\ &\quad - \left(\frac{2\sqrt{\delta}}{1-\delta}\right)^2 (1-x^2)g_n(x) - b_n(g_n(x)-g_{n-2}(x)) + b_{n+1}(g_{n+2}(x)-g_n(x)), \end{aligned} \tag{A.13}$$

$$\begin{aligned} Y_n(x) &= \left(\frac{1+\delta}{1-\delta}x - a_n\right)(g_{n+1}(x)-g_{n-1}(x)) + (x-a_{n-1})g_{n-2}(x) \\ &\quad - \left(\frac{1+\delta}{1-\delta}x - a_{n-1}\right)g_{n-1}(x) - (x-a_{n+1})g_{n+2}(x) + \left(\frac{1+\delta}{1-\delta}x - a_{n+1}\right)g_{n+1}(x). \end{aligned} \tag{A.14}$$

It is easy to see $X_n(x) = Y_n(x) = 0$ by using the explicit forms of a_n, b_n and $g_n(x)$.

As the third step, we show the following:

Proposition A.3. *The polynomial defined by the three-term recurrence (A.1) satisfies the functional equations (A.2)–(A.4).*

Therefore we obtain the following:

Proposition A.4. *The polynomial defined by the functional equations (A.2)–(A.4) satisfies the three-term recurrence (A.1).*

References

[1] Calogero F 1971 Solution of the one-dimensional *N*-body problem with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419

[2] Sutherland B 1972 Exact results for a quantum many-body problem in one-dimension. II *Phys. Rev.* A **5** 1372–6

[3] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220

Moser J 1975 Integrable systems of non-linear evolution equations *Dynamical Systems, Theory and Applications* ed J Moser *(Lecture Notes in Physics* vol 38*)* (Berlin: Springer)

Calogero F, Marchioro C and Ragnisco O 1975 Exact solution of the classical and quantal one-dimensional many body problems with the two body potential $V_a(x) = g^2a^2/\sinh^2 ax$ *Lett. Nuovo Cimento* **13** 383–7

Calogero F 1975 Exactly solvable one-dimensional many body problems *Lett. Nuovo Cimento* **13** 411–6

[4] Corrigan E and Sasaki R 2002 Quantum vs classical integrability in Calogero–Moser systems *J. Phys. A: Math. Gen.* **35** 7017–62

[5] Ragnisco O and Sasaki R 2004 Quantum vs classical integrability in Ruijsenaars–Schneider systems *J. Phys. A: Math. Gen.* **37** 469–79

Sasaki R 2003 Quantum vs classical integrability in Ruijsenaars–Schneider and Calogero–Moser systems *Czech. J. Phys.* **53** 1111–8

[6] M Ruijsenaars S N and Schneider H 1986 A new class of integrable systems and its relation to solitons *Ann. Phys.* **170** 370–405

M Ruijsenaars S N 1987 Complete integrability of relativistic Calogero–Moser systems and elliptic function identities *Commun. Math. Phys.* **110** 191–213

[7] van Diejen J F 1995 The relativistic Calogero model in an external field *Preprint* solv-int/9509002

van Diejen J F 1995 Multivariable continuous Hahn and Wilson polynomials related to integrable difference systems *J. Phys. A: Math. Gen.* **28** L369–74

[8] van Diejen J F 1995 Difference Calogero–Moser systems and finite Toda chains *J. Math. Phys.* **36** 1299–323

[9] van Diejen J F 1994 Integrability of difference Calogero–Moser systems *J. Math. Phys.* **35** 2983–3004

[10] Calogero F 1977 On the zeros of the classical polynomials *Lett. Nuovo Cimento* **19** 505–7

Calogero F 1977 Equilibrium configuration of one-dimensional many-body problems with quadratic and inverse quadratic pair potentials *Lett. Nuovo Cimento* **22** 251–3

[11] Odake S and Sasaki R 2002 Polynomials associated with equilibrium positions in Calogero–Moser systems *J. Phys. A: Math. Gen.* **35** 8283–314

[12] Mimachi K and Yamada Y 1995 Singular vectors of the Virasoro algebra in terms of Jack symmetric polynomials *Commun. Math. Phys.* **174** 447–55

Awata H, Matsuo Y, Odake S and Shiraishi J 1995 Excited states of Calogero-Sutherland model and singular vectors of the W_N algebra *Nucl. Phys.* B **449** 347–74

[13] Stanley R P 1989 Some combinatorial properties of Jack symmetric functions *Adv. Math.* **77** 76–115

[14] Shiraishi J, Kubo H, Awata H and Odake S 1996 A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions *Lett. Math. Phys.* **38** 33–51

Awata H, Kubo H, Odake S and Shiraishi J 1996 Quantum $\mathcal{W}_N$ algebras and Macdonald polynomials *Commun. Math. Phys.* **179** 401–16

[15] Andrews G E, Askey R and Roy R 1999 Special functions *Encyclopedia of Mathematics and its Applications* (Cambridge: Cambridge University Press)

[16] Koekoek R and Swarttouw R F 1996 The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue *Preprint* math.CA/9602214

[17] Bordner A J, Manton N S and Sasaki R 2000 Calogero–Moser models V: supersymmetry and quantum lax pair *Prog. Theor. Phys.* **103** 463–87

Khastgir S P, Pocklington A J and Sasaki R 2000 Quantum Calogero–Moser models: integrability for all root systems *J. Phys. A: Math. Gen.* **33** 9033–64

[18] Szegö G 1939 Orthogonal polynomials *Am. Math. Soc. New York*

[19] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)

[20] Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach)

[21] Jing S and Yang W 2002 A new kind of deformed Hermite polynomials and its application *Preprint* math-ph/0212011